FOR THE IB DIPLOMA PROGRAMME

Mathematics

APPLICATIONS AND INTERPRETATION SL

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WORKED SOLUTIONS

DYNAMIC LEARNING
HODDER EDUCATION

1 Core: Exponents and logarithms

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 1A

44 $\frac{4x^2+8x^3}{2x} = 2x + 4x^2$

45 $\frac{(2x^2y)^3}{8xy} = \frac{8x^6y^3}{8xy} = x^5y^2$

46 $(2ab^{-2})^{-3} = \left(\frac{2a}{b^2}\right)^{-3} = \left(\frac{b^2}{2a}\right)^3 = \frac{b^6}{8a^3}$

47 a $D = \frac{10^6}{n^2}$ so $n^2 = \frac{10^6}{D}$

Then $n = \sqrt{\frac{10^6}{d}} = \frac{1000}{\sqrt{D}}$

b If $n = 2$ then $D = \frac{10^6}{4} = 250\,000$

c If $D = 10^4$ then $n = \frac{10^3}{\sqrt{10^4}} = 10$ so \$10 million should be spent.

48 a When $n = 5, T_A = 1000 = k_A \times 5^3$ so $k_A = \frac{1000}{5^3} = 8$

When $n = 5, T_B = 1000 = k_B \times 5^2$ so $k_B = \frac{1000}{5^2} = 40$

b $\frac{T_A}{T_B} = \frac{k_A n^3}{k_B n^2} = \frac{k_A}{k_B} n$

c Method B will be faster at factorising a 10 digit number.

The answer to part **b** shows that the ratio of times is proportional to n. Since the times are equal for $n = 5$, it follows that method B is faster for $n > 5$ and method A is faster for $n < 5$.

49 $10 + 2 \times 2^x = 18$

$2^{x+1} = 8 = 2^3$

$x = 2$

50 $9^x = 3^{x+5}$

$3^{2x} = 3^{x+5}$

$2x = x + 5$ so $x = 5$

51 $5^{x+1} = 25 \times 5^{2x}$

$5^{x+1} = 5^2 \times 5^{2x} = 5^{2+2x}$

$x + 1 = 2 + 2x$ so $x = -1$

52 $8^x = 2 \times 4^{2x}$

$(2^3)^x = 2 \times (2^2)^{2x}$

$2^{3x} = 2^{4x+1}$

$3x = 4x + 1$ so $x = -1$

53 $25^{2x+4} = 125 \times 5^{x-1}$

$(5^2)^{2x+4} = 5^3 \times 5^{x-1}$

$5^{4x+8} = 5^{x+2}$

$4x + 8 = x + 2$

$3x = -6$ so $x = -2$

54 a $P = 0.8T$ and $R = 5P^2$

Then $R = 5 \times (0.8T)^2 = 5 \times 0.64T^2 = 3.2T^2$

b Rearranging: $T = \sqrt{\frac{R}{3.2}}$

When $R = 2 \times 10^5$ then $T = \sqrt{\frac{2\times10^5}{3.2}} = \sqrt{\frac{10^5}{1.6}} = \sqrt{\frac{10^6}{16}} = \frac{10^3}{4} = 250$ °K

55 a Time $t = \frac{\text{Distance}}{\text{Speed}} = \frac{3}{v}$

b Fuel used $= 0.5v^2 \times t = 0.5v^2 \times \frac{3}{v} = 1.5v$

c If it uses the full tank of fuel, $1.5v = 60$ so $v = 40$.

The maximum constant speed for the journey is 40 km h^{-1}

56

$$\begin{cases} 8^x 2^y = 1 & (1) \\ \frac{4^x}{2^y} = 32 & (2) \end{cases}$$

$(1): (2^3)^x \times 2^y = 1$

$2^{3x+y} = 2^0$

$3x + y = 0 \qquad (3)$

$(2): \frac{(2^2)^x}{2^y} = 32$

$2^{2x-y} = 2^5$

$2x - y = 5 \qquad (4)$

$(3) + (4): 5x = 5$ so $x = 1$ and then $y = -3$

57 $6^x = 81 \times 2^x$

$2^x \times 3^x = 3^4 \times 2^x$

$3^x = 3^4$ so $x = 4$

58 $32 + 2^{x-1} = 2^x$

$32 + 2^{x-1} = 2 \times 2^{x-1}$

$32 = 2^{x-1}$

$2^5 = 2^{x-1}$ so $x = 6$

59 $(x-2)^{x+5} = 1$

So either

(a) $x + 5 = 0$ or

(b) $x - 2 = 1$ or

(c) $x - 2 = -1$ and $x + 5$ is even

Then the solutions are (a) $x = -5$ or (b) $x = 3$ or (c) $x = 1$

60 $2^7 = 128$ and $5^3 = 125$

So $2^7 > 5^3$, so it follows that $(2^7)^{1000} > (5^3)^{1000}$

$2^{7000} > 5^{3000}$.

61 Any integer ending in 6 will have all positive integer powers also ending in 6, and any integer ending in 1 will have all positive integer powers also ending in 1.

It follows that $316^a + 631^b$ will terminate in 7 for any positive integers a and b.

Exercise 1B

14

$$\begin{aligned} a \times b &= (4 \times 10^6) \times (5 \times 10^{-3}) \\ &= (4 \times 5) \times (10^6 \times 10^{-3}) \\ &= 20 \times 10^3 \\ &= 2 \times 10^4 \end{aligned}$$

15

$$\begin{aligned} c \times d &= (1.4 \times 10^3) \times (5 \times 10^8) \\ &= (1.4 \times 5) \times (10^3 \times 10^8) \\ &= 7 \times 10^{11} \end{aligned}$$

16

$$\begin{aligned} \frac{a}{b} &= (4 \times 10^6) \div (5 \times 10^{-3}) \\ &= (4 \div 5) \times (10^6 \div 10^{-3}) \\ &= 0.8 \times 10^9 \\ &= 8 \times 10^8 \end{aligned}$$

17

$$\begin{aligned} c \times d &= (1.4 \times 10^3) \div (2 \times 10^8) \\ &= (1.4 \div 2) \times (10^3 \div 10^8) \\ &= 0.7 \times 10^{-5} \\ &= 7 \times 10^{-6} \end{aligned}$$

18

$$\begin{aligned} a-b &= (4.7\times10^6)-(7.1\times10^5)\\ &= (4.7\times10^6)-(0.71\times10^6)\\ &= 3.99\times10^6 \end{aligned}$$

19

$$\begin{aligned} d-c &= (4.2\times10^{14})-(3.98\times10^{13})\\ &= (4.2\times10^{14})-(0.398\times10^{14})\\ &= 3.802\times10^{14} \end{aligned}$$

20 a $p = 1.22\times10^8$

b

$$\begin{aligned} \frac{p}{q} &= (12.2\times\ 10^7)\div(3.05\times10^5)\\ &= (12.2\div3.05)\times(10^7\div10^5)\\ &= 4\times10^2\\ &= 400 \end{aligned}$$

c $\frac{p}{q} = 4\times10^2$

21 $\frac{6\times10^{23}}{10^{80}} =\ 6\times10^{-57}$

22 $\frac{12\text{ g}}{6.02\times10^{23}} \approx 1.99\times10^{-23}$ g

23 a 15×10^{-15} m $= 1.5\times10^{-14}$ m

b

$$\begin{aligned} V &= \frac{1}{6}\pi(1.5\times10^{-14})^3\\ &= \frac{1}{6}\pi\times3.375\times10^{-42}\\ &= 1.77\times10^{-42}\text{ m}^3 \end{aligned}$$

24 a $(3.04\times10^{13})\div(1.02\times10^{13}) = (3.04\div1.02)\times(10^{13}\div10^{13}) = 2.98$

b $741\times10^6 = 7.41\times10^8$

c Europe:

$$\begin{aligned} \text{Population per m}^2 &= (7.41\times10^8)\div(1.02\times10^{13})\\ &= (7.41\div1.02)\times(10^8\div10^{13})\\ &= 7.26\times10^{-5} \end{aligned}$$

Africa:

$$\begin{aligned} \text{Population per m}^2 &= (1.2\times10^9)\div(3.04\times10^{13})\\ &= (1.2\div3.04)\times(10^9\div10^{13})\\ &= 0.395\times10^{-4}\\ &= 3.95\times10^{-5} \end{aligned}$$

Europe has more population per square metre.

25

$$\begin{aligned} c \times 10^d &= (3 \times 10^a) \times (5 \times 10^b) \\ &= (3 \times 5) \times (10^a \times 10^b) \\ &= 15 \times 10^{a+b} \\ &= 1.5 \times 10^{a+b+1} \end{aligned}$$

a $c = 1.5$

b $d = a + b + 1$

26

$$\begin{aligned} c \times 10^d &= (2 \times 10^a) \div (5 \times 10^b) \\ &= (2 \div 5) \times (10^a \div 10^b) \\ &= 0.4 \times 10^{a-b} \\ &= 4 \times 10^{a-b-1} \end{aligned}$$

a $c = 4$

b $d = a - b - 1$

27

$$\begin{aligned} c \times 10^r &= xy \\ &= (a \times 10^p) \times (b \times 10^q) \\ &= (a \times b) \times (10^p \times 10^q) \\ &= ab \times 10^{p+q} \end{aligned}$$

If $4 < a < b < 9$ then $16 < ab < 81$ so $1.6 < \frac{ab}{10} < 8.1$ which would be the required form for the value to be in standard form.

$$c \times 10^r = \left(\frac{ab}{10}\right) \times 10^{p+q+1}$$

Then $r = p + q + 1$

Exercise 1C

37

$$\begin{aligned} 1 + 2\log_{10} x &= 9 \\ 2\log_{10} x &= 8 \\ \log_{10} x &= 4 \\ x &= 10^4 = 10000 \end{aligned}$$

38

$$\begin{aligned} \log_2(3x + 4) &= 3 \\ 3x + 4 &= 10^3 = 1000 \\ x &= 332 \end{aligned}$$

39 $\ln(e^a e^b) = \ln(e^{a+b}) = a + b$

40

$$\begin{aligned} 10^x &= 5 \\ x &= \log_{10}(5) = 0.699 \end{aligned}$$

41

$$3 \times 10^x = 20$$
$$10^x = \frac{20}{3}$$
$$x = \log_{10}\left(\frac{20}{3}\right) = 0.824$$

42

$$2 \times 10^x + 6 = 20$$
$$2 \times 10^x = 14$$
$$10^x = 7$$
$$x = \log_{10}(7) = 0.845$$

43

$$5 \times 20^x = 8 \times 2^x$$
$$5 \times (2 \times 10)^x = 8 \times 2^x$$
$$5 \times 2^x \times 10^x = 8 \times 2^x$$
$$5 \times 10^x = 8$$
$$10^x = \frac{8}{5} = 1.6$$
$$x = \log_{10} 1.6$$

44 a $\text{pH} = -\log(2.5 \times 10^{-8}) = 7.60$

b $1.9 = -\log[H^+]$

$[\text{H}^+] = 10^{-1.9} = 0.0126$ moles per litre.

45 a When $t = 0$, $R = 10 \times \text{e}^0 = 10$

b When $R = 5$, $5 = 10 \times \text{e}^{-0.1t}$

$$\frac{5}{10} = e^{-0.1t}$$
$$\frac{10}{5} = 2 = e^{0.1t}$$
$$0.1t = \ln 2$$
$$t = 10 \times \ln 2 = 6.93$$

46 a **i** When $t = 0$, $B = 1000 \times \text{e}^0 = 1000$

ii When $t = 2$, $B = 1000 \times \text{e}^{0.2} = 1221$

b When $B = 3000$, $3000 = 1000 \times \text{e}^{0.1t}$

$\text{e}^{0.1t} = 3$

$t = 10 \ln 3 \approx 11.0$ hours

47 Require $200 \times \text{e}^{0.1t} = 100 \times \text{e}^{0.25t}$

$\text{e}^{0.15t} = 2$

$$t = \frac{1}{0.15} \ln 2 \approx 4.62$$

48 a

$$L = 10\log(10^{12} \times 5 \times 10^{-7})$$
$$= 10\log(5 \times 10^{5})$$
$$= 57.0 \text{ db}$$

b

$$L = 10\log(10^{12} \times 5 \times 10^{-6})$$
$$= 10\log(5 \times 10^{6})$$
$$= 67.0 \text{ db}$$

c Increasing I by a factor of 10 Increases the noise level by 10 db

d $90 = 10\log(10^{12}I)$

$$\log(10^{12}I) = 9$$
$$10^{12}I = 10^{9}$$
$$I = 10^{-3} \text{ W m}^{-2}$$

49 a $20 = \mathrm{e}^{\ln 20}$

b $20^x = 7$

$$\left(\mathrm{e}^{\ln 20}\right)^x = \mathrm{e}^{\ln 7}$$
$$\mathrm{e}^{x\ln 20} = \mathrm{e}^{\ln 7}$$
$$x\ln 20 = \ln 7$$
$$x = \frac{\ln 7}{\ln 20}$$

50

$$\begin{cases} \log(xy) = 3 & (1) \\ \log\left(\frac{x}{y}\right) = 1 & (2) \end{cases}$$

$(1): xy = 10^3 \qquad (3)$

$(2): \frac{x}{y} = 10^1 \qquad (4)$

$(3) \times (4): x^2 = 10^4$ so $x = \pm 10^2 = \pm 100$

The solutions are: $x = 100, y = 10$ or $x = -100, y = -10$

51

Tip: You can solve this question very quickly if you know the rule of logarithms that

$log\ a - log\ b = log\left(\frac{a}{b}\right)$

Then $log(10x) - log(x) = log(10) = 1$.

Without this information, the question can still be solved, but it is more time-consuming.

One approach is shown below. If you generalise this, you can prove the rule of logarithms above.

Suppose $\log(10x) - \log x = c$

Then

$$10^c = 10^{\log(10x) - \log x}$$

$$\begin{aligned} &= 10^{\log(10x)} \div 10^{\log x} \\ &= 10x \div x \\ &= 10 \\ &= 10^1 \end{aligned}$$

So $c = 1$

Mixed Practice

1 **a**

$$\begin{aligned} \text{Perimeter} &= 2 \times (2680 \text{ cm} + 1970 \text{ cm}) \\ &= 9300 \text{ cm} \\ &= 9.3 \times 10^3 \text{ cm} \end{aligned}$$

b

$$\begin{aligned} \text{Area} &= (2680 \text{ cm} \times 1970 \text{ cm}) \\ &= 5\,279\,600 \text{ cm}^2 \\ &\approx 5\,280\,000 \text{ cm}^2 \end{aligned}$$

2

$$\begin{aligned} \frac{(3xy^2)^2}{(xy)^3} &= \frac{9x^2y^4}{x^3y^3} \\ &= \frac{9y}{x} \end{aligned}$$

3

$$\begin{aligned} (3x^2y^{-3})^{-2} &= \left(\frac{3x^2}{y^3}\right)^{-2} \\ &= \left(\frac{y^3}{3x^2}\right)^2 \\ &= \frac{y^6}{9x^4} \end{aligned}$$

4 **a** $k = PR = 4\,000\,000$

b When $R = 4$, $P = \frac{k}{4} = 1\,000\,000$

c $R = kP^{-1}$ so when $P = 250\,000$, $R = 16$

5

$$\begin{aligned} 8^x &= 2^{x+6} \\ (2^3)^x &= 2^x \times 2^6 \\ 2^{3x} &= 2^x \times 2^6 \\ 2^{2x} &= 2^6 \end{aligned}$$

$2x = 6$ so $x = 3$

6

$$\begin{aligned} ab &= (3 \times 10^8) \times (4 \times 10^4) \\ &= (3 \times 4) \times (10^8 \times 10^4) \\ &= 12 \times 10^{12} \\ &= 1.2 \times 10^{13} \end{aligned}$$

7

$$\begin{aligned}\frac{a}{b} &= (1\times10^9)\div(5\times10^{-4})\\ &= (1\div5)\times(10^9\div10^{-4})\\ &= 0.2\times10^{13}\\ &= 2\times10^{12}\end{aligned}$$

8

$$\begin{aligned}a-b &= (5\times10^5)-(3\times10^4)\\ &= (5\times10^5)-(0.3\times10^5)\\ &= 4.7\times10^5\end{aligned}$$

9

$$\begin{aligned}\text{Time} &= \frac{\text{distance}}{\text{speed}}\\ &= (1.5\times10^{11}\text{ m})\div(3\times10^8\text{ m s}^{-1})\\ &= (1.5\div3)\times(10^{11}\div10^8)\text{ s}\\ &= 0.5\times10^3\text{ s}\\ &= 500\text{ s}\end{aligned}$$

10

$$\begin{aligned}\log(x+1) &= 2\\ x+1 &= 10^2 = 100\\ x &= 99\end{aligned}$$

11

$$\begin{aligned}\ln(2x) &= 3\\ 2x &= \mathrm{e}^3\\ x &= 0.5\mathrm{e}^3\end{aligned}$$

12 $\mathrm{e}^x = 2$

$x = \ln 2 = 0.693$ (to 3 s.f.)

13 $5\times10^x = 17$

$10^x = 3.4$

$x = \log 3.4 = 0.531$ (to 3 s.f.)

14 $5\mathrm{e}^x - 1 = y$

$$\begin{aligned}5\mathrm{e}^x &= y+1\\ \mathrm{e}^x &= \frac{y+1}{5}\\ x &= \ln\left(\frac{y+1}{5}\right)\end{aligned}$$

15 a $2^m = 8 = 2^3$ so $m = 3$

$2^n = 16 = 2^4$ so $n = 4$

b

$$\begin{aligned} 8^{2x+1} &= 16^{2x-3} \\ (2^3)^{2x+1} &= (2^4)^{2x-3} \\ 2^{6x+3} &= 2^{8x-12} \\ 6x+3 &= 8x-12 \\ 2x &= 15 \\ x &= 7.5 \end{aligned}$$

16

$$\begin{aligned} (7\times 10^a)\times(4\times 10^b) &= c\times 10^d \\ (7\times 4)\times(10^a\times 10^b) &= c\times 10^d \\ 28\times 10^{a+b} &= c\times 10^d \\ 2.8\times 10^{a+b+1} &= c\times 10^d \end{aligned}$$

a $c = 2.8$

b $d = a + b + 1$

17

$$\begin{aligned} (6\times 10^a)\div(5\times 10^b) &= c\times 10^d \\ (6\div 5)\times(10^a\div 10^b) &= c\times 10^d \\ 1.2\times 10^{a-b} &= c\times 10^d \end{aligned}$$

a $c = 1.2$

b $d = a - b$

18 $\mathrm{pH} = 6.1 + \log\left(\frac{[\mathrm{HCO_3^-}]}{[\mathrm{H_2CO_3}]}\right)$

Rearranging:

$$\frac{[\mathrm{HCO_3^-}]}{[\mathrm{H_2CO_3}]} = 10^{\mathrm{pH}-6.1}$$

$$[\mathrm{H_2CO_3}] = \frac{[\mathrm{HCO_3^-}]}{10^{\mathrm{pH}-6.1}}$$

When $\mathrm{pH} = 7.35$ and $[\mathrm{HCO_3^-}] = 0.579$, $[\mathrm{H_2CO_3}] = 0.0326$

When $\mathrm{pH} = 7.45$ and $[\mathrm{HCO_3^-}] = 0.579$, $[\mathrm{H_2CO_3}] = 0.0259$

So the range of carbonic acid concentrations is 0.0259 to 0.0326

19 $v = 1350(1 - \mathrm{e}^{-0.007t})$

a When $t = 1$, $v = 1350\times(1-\mathrm{e}^{-0.007}) = 9.42\ \mathrm{m\ s^{-1}}$

b At $t = 600$, the model predicts $v = 1350\times(1-\mathrm{e}^{-4.2}) = 1330\ \mathrm{m\ s^{-1}}$

Yes, he would be expected to break the speed of sound.

Ordinarily, air resistance would prevent such high speeds being reached so soon during a free fall, but by jumping from the edge of the atmosphere, the effect of air resistance was reduced during the initial period of the fall.

20 **a** When $A = 1000$, $S = \log 1000 = 3$

b $A' = 10A$

$S' = \log 10A$

$S' = \log 10 + \log A$

$S' = 1 + S$

When A increases by a factor of 10, S increases by one unit.

c $A = 10^S$ so when $S = 9.5$, $A = 10^{9.5}$ μm $= 10^{3.5}$ m $= 3162$ m

21 $3 \times 20^x = 2^{x+1}$

$$3 \times 20^x = 2 \times 2^x$$

$$10^x = \frac{2}{3}$$

$$x = \log\left(\frac{2}{3}\right) \approx -0.176$$

22

$$\begin{cases} 9^x \times 3^y = 1 \\ \dfrac{4^x}{2^y} = 16 \end{cases}$$

$$\begin{cases} 3^{2x+y} = 3^0 \\ 2^{2x-y} = 2^4 \end{cases}$$

$$\begin{cases} 2x + y = 0 & (1) \\ 2x - y = 4 & (2) \end{cases}$$

$(1) + (2)$: $4x = 4$ so $x = 1, y = -2$

23

$$\begin{cases} \log(xy) = 0 \\ \log\left(\dfrac{x^2}{y}\right) = 3 \end{cases}$$

$$\begin{cases} xy = 1 & (1) \\ \dfrac{x^2}{y} = 10^3 & (2) \end{cases}$$

$(1) \times (2)$: $x^3 = 10^3$ so $x = 10, y = 0.1$

2 Core: Sequences

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 2A

21 $u_1 = 7, d = 11$

a $u_{20} = u_1 + 19d = 216$

b $S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$

$$S_{20} = \frac{20}{2}(2 \times 7 + 11 \times 19) = 2230$$

22 $u_1 = 3, u_2 = 7 = u_1 + d$

a $d = 4$

b $u_8 = u_1 + 7d = 31$

c $S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$

$$S_{15} = \frac{15}{2}(2 \times 3 + 4 \times 14) = 465$$

23 $d = 5, u_2 = 13 = u_1 + d$

a $u_1 = 8$

b $S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$

$$S_{10} = \frac{10}{2}(2 \times 8 + 5 \times 9) = 305$$

24 $u_1 = -8, u_{16} = 67 = u_1 + 15d$

a $15d = 67 - u_1 = 75$ so $d = 5$

b $u_{25} = u_1 + 24d = 112$

25 **a** 4% of £300 is £300 $\times$ 0.04 = £12

Simple interest: The same interest sum is earned each year.

At the end of the first year, Sam has £300 + 1 $\times$ £12 = £312.

b At the end of the tenth year, Sam has £300 + 10 $\times$ £12 = £420.

26 On his 21st birthday, the grandparents have paid in 20 supplementary £10 amounts.

The balance is £100 + 20 $\times$ £10 = £300

27 The nth stair is u_n cm off the ground.

Then $u_1 = 10$ and $d = 20$.

$$u_n = u_1 + d(n-1)$$
$$10 + 20(n-1) = 270$$
$$n - 1 = \frac{260}{20} = 13$$
$$n = 14$$

28 $u_1 = 11, u_n = 75 = u_1 + d(n-1)$

a If $d = 8$ then $8(n-1) + 11 = 75$ so $n - 1 = \frac{64}{8} = 8; n = 9$

$$S_n = \frac{n}{2}(u_1 + u_n) = \frac{9}{2}(11 + 75) = 387$$

b If $d = 4$ then $4(n-1) + 11 = 75$ so $n - 1 = \frac{64}{4} = 16; n = 17$

$$S_n = \frac{n}{2}(u_1 + u_n) = \frac{17}{2}(11 + 75) = 731$$

Tip: Be careful not to assume that halving the difference for a sequence with a fixed start and end point either doubles the number of terms (it doubles $n - 1$ instead!) or doubles the total sum. Neither is valid.

29 $u_{10} = 26 = u_1 + 9d$ (1)

$u_{30} = 83 = u_1 + 29d$ (2)

$(2) - (1): 20d = 57$

$u_{50} = u_1 + 49d = u_{30} + 20d = 140$

Tip: Be alert to simple approaches. At no point in this problem did you need to calculate u_1 or d.

30 a $u_1 = 8, d = 3, u_n = 68 = u_1 + d(n-1)$

Then $3(n-1) = 60$ so,

$n - 1 = 20$

$n = 21$

b $S_n = \frac{n}{2}(2u_1 + d(n-1))$

$$S_{21} = \frac{21}{2}(2 \times 8 + 3 \times 20) = 798$$

31 a $u_7 = 35 = u_1 + 6d$ (1)

$u_{18} = 112 = u_1 + 17d$ (2)

$(2) - (1): 11d = 77$

So $d = 7$, and then $u_1 = u_7 - 6d = -7$

b $S_n = \frac{n}{2}(2u_1 + d(n-1))$

$$S_{18} = \frac{18}{2}(2 \times (-7) + 7 \times 17) = 945$$

32 **a** $u_{10} = 16 = u_1 + 9d \qquad (1)$

$u_{30} = 156 = u_1 + 29d \qquad (2)$

$(2) - (1): 20d = 140$

$u_{50} = u_1 + 49d = u_{30} + 20d = 296$

b $d = 7$ so $u_1 = 16 - 9d = -47$

$$S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$$

$$s_{20} = \frac{20}{2}(2 \times (-47) + 7 \times 19) = 390$$

33 Let $u_n = 5n - 3$

Then $u_1 = 2$ and $d = 5$

$$S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$$

$$S_{16} = \frac{16}{2}(2 \times 2 + 5 \times 15) = 632$$

34 There must be a common difference between the terms.

So:

$$(2x+1) - (3x+1) = (4x-5) - (2x+1)$$
$$-x = 2x - 6$$
$$3x = 6$$
$$x = 2$$

35 $u_5 = 2u_2, u_7 = 28$

Then $u_1 + 4d = 2(u_1 + d) \qquad (1)$

and $u_1 + 6d = 28 \qquad (2)$

$(1): u_1 = 2d$

Substituting into $(2): 8d = 28$ so $2d = 7$

$u_{11} = u_7 + 4d = 28 + 2 \times 7 = 42$

36 $u_1 + u_2 + u_3 = u_{10}, u_7 = 27$

$u_1 + (u_1 + d) + (u_1 + 2d) = u_1 + 9d \qquad (1)$

$u_1 + 6d = 27 \qquad (2)$

$(1): 3u_1 + 3d = u_1 + 9d$ so $2u_1 = 6d$

Substituting into $(2): 3u_1 = 27$ so $u_1 = 9$ and then $d = 3$

Then $u_{12} = u_1 + 11d = 42$

37 Arithmetic sequence with common difference $d = 4$

$$S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$$

$$S_{20} = \frac{20}{2}(2 \times 10 + 4 \times 19) = 960$$

38 **a** $u_1 = 7, d_u = 12$

If the sequence has N terms, $u_N = 139$

$u_N = u_1 + d_u(N-1) = 7 + 12(N-1) = 139$

$12N - 5 = 139$

$N = 12$

$$S_N = \frac{N}{2}(u_1 + u_N) = \frac{12}{2}(7 + 139) = 876$$

b $v_1 = 7, d_v = 6$

If the sequence has M terms, $v_M = 139$.

$$\begin{aligned} v_M &= v_1 + d_v(M-1) \\ &= 7 + 6(M-1) = 139 \end{aligned}$$

$6M + 1 = 139$

$M = 23$

$$S_M = \frac{M}{2}(v_1 + v_M) = \frac{23}{2}(7 + 139) = 1679$$

c $w_1 = 7, d_w = 6, N = 12$

$$S_N = \frac{N}{2}\left(2w_1 + d_w(N-1)\right) = \frac{12}{2}(2 \times 7 + 6 \times 11) = 480$$

39 Estimating $u_1 = 53, u_5 = 211 = u_1 + 4d$ so $d = \frac{158}{4} = 39.5$

Then $u_{10} = u_1 + 9d = 408.5$

40 Estimating $u_1 = 3, u_4 = 33 = u_1 + 3d$ so $d = \frac{30}{3} = 10$

Then $u_6 = u_1 + 5d = 53$

41 $u_1 = 1$

$u_2 = a = u_1 + d$ (1)

$u_3 = 3a + 5 = u_1 + 2d$ (2)

$2(1) - (2): u_1 = -a - 5$

So $a - 5 = 1$ and therefore $a = -6$ so $d = -7$

Then $u_4 = u_1 + 3d = -20$

Tip: Alternatively, use the fact that $u_n + 1 = 2u_n - u_{n-1}$ for any n in an arithmetic sequence.

So $u_3 = 2u_2 - u_1$ so $3a + 5 = 2a - 1$ from which $a = -6$

Then $u_4 = 2u_3 - u_2 = 5a + 10 = -20$ directly, without calculating d at all.

42 Let u_n be the number of minutes of screentime on day n.

$u_1 = 200$ and $d = -5$ in the arithmetic progression.

a He gives gets to zero minutes on day N where $u_N = 0$

$u_N = u_1 + d(N-1) = 200 - 5(N-1) = 0$ so $N = 41$

b

$$S_N = \frac{N}{2}\left(2u_1 + d(N-1)\right)$$
$$= \frac{41}{1}(2 \times 200 - 5 \times 41) = 7995 \text{ minutes total}$$

43 Let u_n be the number of steps taken on day n.

$u_1 = 1000, d = 500$ in an arithmetic sequence.

a Require least N such that $u_N = 10000$

$$u_N = u_1 + (N-1)d$$
$$10000 = 1000 + 500(N-1)$$
$$N = 1 + \frac{9000}{500} = 19$$

On the 19th day he takes 10 000 steps.

b Require M such that $S_M = 540\,000$

$$S_M = \frac{M}{2}\left(2u_1 + d(M-1)\right)$$
$$540000 = \frac{M}{2}\left(2 \times 1000 + 500(M-1)\right)$$
$$1080000 = 2000M + 500M^2 - 500M$$
$$2160 = M^2 + 3M$$
$$M^2 + 3M - 2160 = 0$$
$$(M+48)(M-45) = 0$$

$M = -48$ (reject) or $M = 45$

It takes 45 days.

44

Comment: Three methods are shown here; the first method takes the formula given for the sum and compares it to the standard general formula to find u_1 and d.

The second method is uses the fact that $u_1 = S_1$ and that the difference in sequential sums is a single term: $S_N - S_{N-1} = u_N$; from this we can find both u_1 and d indirectly, if needed.

The third uses again that $S_N - S_{N-1} = u_N$ to find the formula for the general term u_N. For this question, this is unnecessarily laborious but is potentially useful in more complicated situations, or when only this is required, as in question 25 below.

Method 1: Compare formulae to find parameter values

a $S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right) \equiv 2n^2 + n$ for all n for this sequence

$$n(2u_1 - d + dn) \equiv 4n^2 + 2n$$
$$dn^2 + (2u_1 - d)n \equiv 4n^2 + 2n$$

Comparing coefficients, $d = 4$ and $2u_1 - d = 2$ so $2u_1 = 6$ and so $u_1 = 3$

Then $u_1 = 3, u_2 = 7$

b $u_{50} = u_1 + 49d = 3 + 49 \times 4 = 199$

Method 2: Use given formula directly

a $S_n = 2n^2 + n$ so $S_1 = 3$ and $S_2 = 10$.

$u_1 = S_1 = 3$

$u_2 = S_2 = S_1 = 7$

b

$$\begin{aligned} u_{50} &= S_{50} - S_{49} \\ &= (2 \times 50^2 + 50) - (2 \times 49^2 + 49) \\ &= 5050 - 4851 = 199 \end{aligned}$$

Method 3: Use given sum formula to find the formula for the general term.

$S_n - S_{n-1} = u_n$

$$\begin{aligned} u_n &= (2n^2 + n) - \left(2(n-1)^2 + (n-1)\right) \\ &= 2n^2 + n - (2n^2 - 4n + 2 + n - 1) \\ &= 4n - 1 \end{aligned}$$

a $u_1 = 4 \times 1 - 1 = 3$

$u_2 = 4 \times 2 - 1 = 7$

b $u_{50} = 4 \times 50 - 1 = 199$

45 $S_n - S_{n-1} = u_n$

$$\begin{aligned} u_n &= (n^2 + 4n) - \left((n-1)^2 + 4(n-1)\right) \\ &= n^2 + 4n - (n^2 - 2n + 1 + 4n - 4) \\ &= 2n + 3 \end{aligned}$$

46 $u_9 = 5u_3$

Then $u_1 + 8d = 4(u_1 + 2d)$

$u_1 + 8d = 4u_1 + 8d$

$u_1 = 0$

47 $7 \times 15 = 105$

$7 \times 143 = 1001$

The multiples of 7 form an arithmetic sequence with common difference $d = 7$.

Let $u_1 = 105$, $d = 7$ and $n = 143 - 15 = 128$

$$S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$$

$$S_{128} = \frac{128}{2}(2 \times 105 + 7 \times 127) = 70336$$

48 Multiples of 5 (not including 0 which does not change the sum): $u_1 = 5, d = 5, u_{20} = 100$

$$S_{20} = \frac{20}{2}(2 \times 5 + 5 \times 19) = 1050$$

Multiples of 15 (not including 0): $v_1 = 15, d = 15, u_6 = 90$

$$S_6 = \frac{6}{2}(2 \times 15 + 15 \times 5) = 315$$

Then the sum of multiples of 5 which are not also multiples of 3 will be $1050 - 315 = 735$

49 $u_{10} = 3u_1$ and $S_{10} = 400$ cm

$S_{10} = \frac{10}{2}(u_1 + u_{10}) = 20u_1 = 400$ cm so $u_1 = 20$ cm

50 $u_1 = 4$

$u_2 = a = 4 + d$ (1)

$u_3 = b = 4 + 2d$ (2)

$u_4 = a - b = 4 + 3d$ (3)

$(3) - (2) - (1): -2b = -4$ so $b = 2$. Then $d = -1$ so $u_6 = u_1 + 5d = -1$

51 a

$$\begin{aligned} u_n - u_{n-1} &= (a + nd) - (a + (n-1)d) \\ &= d \end{aligned}$$

The difference between consecutive terms is shown to be constant d.

b

$$\begin{aligned} u_n - u_{n-1} &= (an^2 + bn) - (a(n-1)^2 + b(n-1)) \\ &= (an^2 + bn) - (an^2 - 2an + a + bn - b) \\ &= 2an + a - b \\ &= A + nD \end{aligned}$$

where $A = a - b$ and $D = 2a$

Using part **a**, this is the formula for an arithmetic sequence with constant difference $D = 2a$.

52 a The first 9 positive integers are single digits, for a total of 9 digits written

The next 10 integers (10 to 19) are double-digit numbers, so contain a total of 20 digits.

In all, she has written 29 digits.

b The values up to 99 will contain a total of

9 digits (single digit values 1 to 9)

180 digits (double digit values 10 to 99)

If she has written a total of 342 digits then the remaining 153 digits are from triple-digit numbers.

$153 \div 3 = 51$ so she has written 51 triple-digit values

100 is the first, so 150 is the 51st.

Alessia has written values up to 150.

Exercise 2B

16 a $u_1 = 128, r = 0.5$

$u_8 = u_1 \times r^7 = 1$

b $S_8 = u_1 \times \frac{1-r^8}{1-r} = 128 \times \frac{1-\frac{1}{256}}{1-\frac{1}{2}} = 255$

17 $u_1 = 3, u_2 = 6 = r \times u_1$

a $r = 2$

b $u_6 = u_1 \times r^5 = 96$

c $S_{10} = u_1 \times \frac{r^{10}-1}{r-1} = 3 \times \frac{1024-1}{2-1} = 3069$

18 $u_2 = 24 = u_1 \times r \qquad (1)$

$u_5 = 81 = u_1 \times r^4 \qquad (2)$

a $(2) \div (1): r^3 = \frac{81}{24} = \frac{27}{8} = \left(\frac{3}{2}\right)^3$

$r = 1.5$

b $u_7 = u^5 \times r^2 = 81 \times \frac{9}{4} = 182.25$

19

Tip: Rather than use n to count the number of days, it can be simpler to use n to count the number of complete periods of doubling. You then need less work for calculating terms of the sequence, but must take care when interpreting what day corresponds to what value of n.

As is always the case in problems like this, if the question does not define the terms of a sequence, you should give a definition in your answer, but are free to do this however seems most convenient.

Let u_n be the area of algal cover on day $8n - 7$ (the start of the nth 8-day period).

$u_1 = 15, r = 2$ in a geometric sequence.

The start of week 9 is day 57 which corresponds to $n = 8$

$u_8 = u_1 \times r^7 = 15 \times 2^7 = 1920 \text{ cm}^2$

20 Let u_n be the concentration on day $2n - 1$.

$u_1 = 1.2$ and $r = \frac{1}{2}$ for a geometric sequence.

After 12 days, $n = 6$.

$u_6 = u_1 \times r^5 = 1.2 \times 0.5^5 = 0.0375 \text{ mg ml}^{-1}$

21 $u_1 = 8, u_1 + u_2 = 12$ so $u_2 = 4 = u_1 \times r$

$r = 0.5$

$$S_5 = u_1 \times \frac{1 - r^5}{1 - r} = 8 \times \frac{1 - \frac{1}{32}}{1 - \frac{1}{2}} = 15.5$$

22 Let u_n be the time taken on the nth attempt

$u_1 = 5, r = 0.8$ in a geometric sequence.

$u_{10} = u_1 \times r^9 = 0.671$ seconds

23

Tip: As in several questions towards the end of this exercise, the question describes a situation and defining a sequence should be the first step of a formal answer.

Let u_n be the volume of water at the start of day n of the drought.

a $u_1 = 5000, r = 0.92$ in a geometric sequence

$u_6 = u_1 \times r^4 = 3580 \text{ m}^3$

b Require N such that $u_N < 2000$

Then $u_1 \times r^{N-1} < 2000$

$$r^{N-1} < \frac{2000}{u_1} = 0.4$$

$$(N-1)\log r < \log 0.4$$

$$-0.036(N-1) < -0.398$$

$$N - 1 > \frac{0.398}{0.036}$$

$$N > 11.99$$

At the start of day 12, the reservoir holds less than 2000 m^3, so it takes 11 days to use up 3000 m^3.

24 $u_5 = u_1 \times r^4$ (1)

$u_2 = u_1 \times r$ (2)

But $u_5 = 8u_2$ so $r^3 = 8$, from which $r = 2$

$$\frac{S_8}{u_1} = \frac{r^8 - 1}{r - 1} = \frac{256 - 1}{1} = 255$$

25 Let u_n be the number of grains on the nth square; $\{u_n\}$ is a geometric sequence with $r = 2$.

a $u_{64} = u_1 \times r^{63} = 2^{63} = 9.22 \times 10^{18}$

b $S_{64} = \frac{(r^{64} - 1)}{r - 1} = 2^{64} - 1 = 1.84 \times 10^{19}$

c Mass of Sissa's reward: $s_{64} \times 0.1 \text{ g} = 1.84 \times 10^{18} \text{ g}$

This would take $1.84 \times 10^{18} \div (7.5 \times 10^{14}) \approx 2450$ years.

26 $r = \frac{u_n}{u_{n-1}}$

$\frac{y^3}{xy^2} = yx^{-1}$ and $\frac{x^{-1}y^4}{y^3} = x^{-1}y$ so the sequence is consistent, and no relation between x and y can be determined.

$u_n = u_1 \times r^{n-1}$ so the general term $u_n = xy^2 \times (x^{-1}y)^{n-1} = x^{2-n}y^{n+1}$

27 Geometric sequence with $u_1 = 3$ and $r = 2$.

$$S_{10} = u_1 \frac{(r^{10} - 1)}{r - 1} = 3 \times \frac{2^{10} - 1}{2 - 1} = 3 \times 1023 = 3069$$

28 $r = \frac{u_2}{u_1} = x$ and also $r = \frac{u_3}{u_2} = \frac{2x^2 + x}{x} = 2x + 1$

So $x = 2x + 1$

$x = -1$

$u_{10} = u_1 \times r^9 = -1$

29 Sum of a geometric sequence with $u_1 = 3$ and $r = 3$.

$$S_{10} = u_1 \frac{r^{10} - 1}{r - 1} = 3 \times \frac{3^{10} - 1}{3 - 1} = 88\,572$$

30 Let u_n be the height reached on the nth bounce, in metres.

$u_1 = 0.6$ and $r = 0.8$ in a geometric sequence.

a $u_5 = u_1 \times r^4 = 0.246$ so the height of the 5th bounce is 0.246 m.

b From the top of the 1st bounce to the top of the 5th bounce would be distance

$$u_1 + 2(u_2 + u_3 + u_4) + u_5 = 0.6 + 2(0.48 + 0.384 + 0.3072) + 0.24576$$
$$\approx 3.19 \text{ m}$$

c The bounce height and value of r are only given to 1 significant figure, so even if a geometric model were precise for the loss of energy from a bouncing system, the accuracy of the output would be poor for high powers of r.

In any case, at the level of detail predicted by the model at the 20th bounce, we would expect that measurement error and imperfections in the ground surface and ball surface and composition would swamp the prediction.

Exercise 2C

16 $£800 \times 1.03^4 = £900.41$

17 $£10\,000 \times 1.05^7 = £14\,071$

18 4% annual rate is equivalent to $1 + \frac{0.04}{12} = 1.00333$ monthly multiplier

$\$8000 \times 1.00333^{30} = \$8\,839.90$

19 5.8% annual rate is equivalent to $1 + \frac{0.058}{12} = 1.00483$ monthly multiplier.

$€15\,000 \times 1.00483^{18} = €16\,360.02$

20 $£20\,000 \times 0.85^5 = £8870$ (to 3 s. f.)

21 a The return will be better when interest is applied monthly, due to the effect of compounding.

After 1 year at 6% compounded annually, the account stands at

$£1000 \times 1.06 = £1060$

After 1 year at 6% compounded monthly, the account stands at

$$£1000 \times \left(1 + \frac{0.06}{12}\right)^{12} = £1061.68$$

b After 10 years at 6% compounded annually, the account stands at

$£1000 \times 1.06^{10} = £1790.85$

After 10 years at 6% compounded monthly, the account stands at

$$£1000 \times \left(1 + \frac{0.06}{12}\right)^{120} = £1819.40$$

The difference is £28.55

22 20% depreciation and 2.5% interest rate: After the first year the value is

$£15\,000 \times 0.775 = £11\,625$

10% depreciation and 2.5% interest rate for 4 years: $£11\,625 \times 0.875^4 = £6814.36$

Worked solutions

23

Year	Start-year value ($)	Depreciation expense ($)	End-year value($)
1	20 000	6 000	14 000
2	14 000	4 200	9 800
3	9 800	2 940	6 860
4	6 860	2 058	4 802
5	4 802	1 441	3361
6	3361	1 008	2 353
7	2 353	706	1 647
8	1 647	147	1 500

24 The annual real terms percentage change is $r \approx c - i = 6.2\% - 3.2\% = 3\%$

So over 5 years, the real terms percentage increase is $1.03^5 - 1 = 15.9\%$

25 12% annual interest: 1% monthly interest

Monthly interest of 1% gives $(1 + 0.01)^{12} - 1 = 12.7\%$ annual equivalent rate (AER)

Tip: Be careful to distinguish between 'annual interest rate' which is 12 × monthly interest and "Annual Equivalent Rate" (AER), which is the effective rate over a year, after consideration of compounding.

AER is often published in financial offers because it allows easy comparison between products applying compound interest at different intervals.

For example:

Account *A* compounds monthly at an annual rate of 12% so would have a monthly rate of 1% for an AER of 12.68% (as in this question)

Account *B* compounds quarterly at an annual rate of 12.1% so would have a quarterly rate of 3.025% for an AER of 12.66%. Although account *B* has a higher annual rate, account *A* actually has a slightly higher AER.

26 a Real terms percentage increase $r = \frac{1+\frac{c}{100}}{1+\frac{i}{100}} - 1 = \frac{1.05}{1.025} - 1 = 2.44\%$

b Over the course of 5 years, this would produce a $1.0244^5 - 1 = 12.8\%$ increase in real terms.

27 a $250 \times (1 + 10^9) \approx 250 \times 10^9$ marks (250 billion marks)

b Using the exact formula for adjustment: $r = \frac{1+\frac{c}{100}}{1+\frac{i}{100}} = \frac{1.2}{1+10^9}$

So at year end, 2×10^6 marks would have a real terms value of

$$2 \times 10^6 \times \left(\frac{1.2}{10^9}\right) = 2.4 \times 10^{-3} = 0.0024 \text{ marks}$$

c $= \frac{1+\frac{c}{100}}{1+\frac{i}{100}} = \frac{1.15}{1+10^9}$

So at year end, 25×10^6 marks after application of interest would have a real terms value of $25 \times 10^6 \times \left(\frac{1.15}{10^9}\right) \approx 29 \times 10^{-3} = 0.029$ marks

Mixed Practice

1 **a** 4.5% nominal annual rate is equivalent to $1 + \frac{0.045}{12} = 1.00375$ monthly multiplier

$\$5000 \times 1.00375^{7\times12} = \$6847.26$

b Current value $= 7000$, Future value $= 14\,000$, $n = 10$, payments/year $= 1$.

Calculator gives $I\% = 7.18$

Carla requires at least 7.18% annual rate to achieve her aim.

2 **a** **i** $\{d\}$ is an arithmetic sequence, with common difference -0.05

ii $\{b\}$ is a geometric sequence, with common ratio $\frac{3}{2}$

b **i** Common ratio is $\frac{-3}{-6} = \frac{1}{2}$

ii $e_1 = -6, r = \frac{1}{2}$

$e_{10} = e_1 \times r^9 = -\frac{6}{2^9} = -\frac{3}{256}$

3 **a** $d = u_9 - u_8 = 2$

b $u_8 = u_1 + 7d = u_1 + 14 = 10$ so $u_1 = -4$

c $S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right)$

So $S_{20} = \frac{20}{2}(2 \times (-4) + 2 \times 19) = 300$

4 **a** $u_1 = 2$ and $u_2 = 8 = u_1 \times r$ so $r = 4$

b $u_5 = u_1 \times r^4 = 2 \times 4^4 = 512$

c $S_n = u_1 \frac{(r^n - 1)}{r-1}$

$$S_8 = 2\frac{4^8 - 1}{4 - 1} = 43\,690$$

5 Let u_n be net profit in year n, in thousand dollars.

$u_1 = -100$ and $\{u\}$ follows an arithmetic progression with common difference $d = 15$

Least N such that $u_N > 0$:

$$\begin{aligned} u_N = u_1 + (N-1)d &> 0 \\ -100 + 15(N-1) &> 0 \\ 15(N-1) &> 100 \\ N &> 1 + \frac{100}{15} = 7.6 \end{aligned}$$

The company is first profitable in the 8th year.

6 Let u_n be the value at the start of the nth year, in thousand dollars.

$u_1 = 25$ and $\{u\}$ follows an arithmetic progression with common difference $d = -1.5$

Least N such that $u_N \leqslant 10$:

$$\begin{aligned} u_N = u_1 + (N-1)d &\leqslant 10 \\ 25 - 1.5(N-1) &\leqslant 10 \\ 1.5(N-1) &\geqslant 15 \\ N &\geqslant 11 \end{aligned}$$

At the start of the 11th year, the car has value \$10 000 so it takes 10 years to fall to that value.

7 Let u_n be the height of the sunflower n weeks after being planted, in cm.

$u_1 = 20$ and $\{u\}$ follows a geometric progression with common ratio $r = 1.25$

a $u_5 = u_1 \times r^4 = 20 \times 1.25^4 = 48.8$ cm

b Least N such that $u_N > 100$:

$$\begin{aligned} u_N &= u_1 \times r^{N-1} \\ 20 \times 1.25^{N-1} &> 100 \\ 1.25^{N-1} &> 5 \end{aligned}$$

Tip: If you are comfortable using logarithms with base other than 10 or e then you can calculate the solution directly as $N - 1 > \log_{1.25} 5$. Otherwise, use a change of base method or simply let the calculator solve this directly using an equation solver or graph solver.

$$\begin{aligned} N - 1 &> 7.2 \\ N &> 8.2 \end{aligned}$$

It takes 9 weeks for the plant to exceed 100 cm.

8 **a** Current value $= 500$, $n = 16$, payments/year $= 4$, I% $= 3$.

Calculator gives Future value $= 563.50$

After 4 years, Kunal has a balance of 563.50 euros.

b Current value $= 500$, payments/year $= 4$, I% $= 3$, Future value $= 600$

Calculator gives $n = 24.4$

It will take 25 quarters (six and a quarter years) for Kunal to earn 100 euros of interest.

9 Let u_n be the population at the end of year $2000 + n$

$u_{18} = 7.7 \times 10^9$ and $\{u\}$ follows a geometric progression with $r = 1.011$.

a $u_{22} = u_{18} \times r^4 = 8.04 \times 10^9$

According to the model, at the end of 2022 the world's population will be approximately 8.0 billion.

b Least N such that $u_N > 9 \times 10^9$:

$$\begin{aligned} u_N &= u_{18} \times r^{N-18} \\ 7.7 \times 1.011^{N-1} &> 9 \\ N &> 32.3 \end{aligned}$$

According to the model, the world's population will exceed 9 billion during 2033.

10 **a** **i** $u_2 = 30; u_5 = 90 = u_2 + 3d$ so $d = \frac{90-30}{3} = 20$

ii $u_1 = u_2 - d = 10$

b Let $\{v\}$ be the geometric sequence.

$v_1 = u_1 = 10; v_2 = u_2 = 30; v_3 = u_5 = 90$

So the common ratio $r = 3$

$v_7 = v_1 \times r^6 = 10 \times 3^6 = 7290$

11 **a** Substituting for n in the formula of S_n:

i $S_1 = 6 + 1 = 7$

ii $S_2 = 12 + 4 = 16$

b $u_2 = S_2 - S_1 = 16 - 7 = 9$

c $S_1 = u_1 = 7$ and $u_2 = 9$ so common difference $d = u_2 - u_1 = 2$

d $u_{10} = u_1 + 9d = 7 + 18 = 25$

e Least N such that $u_N > 1000$:

$$u_N = u_1 + d(N - 1)$$
$$7 + 2(N - 1) > 1000$$
$$N > 1 + \frac{993}{2} = 497.5$$

The first term which exceeds 1000 is u_{498}

f $S_n = 1512 = 6n + n^2$

Solving the quadratic: $n = 36$ (reject solution $n = -42$)

12 $u_1 = x$

$u_2 = u_1 + d = 2x + 4$ so $d = x + 4$ (1)

$u_3 = u_1 + 2d = 5x$ so $d = 2x$ (2)

$(1)\&(2)$: $x + 4 = 2x$ so $x = 4$

13 **a** The difference between consecutive terms in the sequence are

$d_1 = 10, d_2 = 12, d_3 = 9, d_4 = 9.$

Mean difference is 10 and all the observed differences are close to this value.

b $u_6 = u_1 + 5d$

Taking the value $d = 10$, this predicts $u_6 = 74$ audience members.

14 Sum of a geometric sequence with $u_1 = 2$ and $r = 2$.

$$S_{12} = u_1\frac{r^{12}-1}{r-1} = 2 \times \frac{2^{12}-1}{2-1} = 8190$$

15 Let u_n be the number of widgets sold in the nth month.

$u_1 = 100$ and $\{u\}$ follows an arithmetic progression with common difference $d = 20$.

$$S_n = \frac{n}{2}\left(2u_1 + d(n - 1)\right)$$

Require least N such that $S_N \geqslant 4000$

$$\frac{N}{2}\big(200 + 20(N-1)\big) \geqslant 4000$$
$$10N^2 + 90N - 4000 \geqslant 0$$

From calculator, this has solution $N \geqslant 16$

It takes 16 months to sell a total of 4000 widgets.

16 $u_1 = a^2b^2$ and $r = \frac{u_2}{u_1} = \frac{a^4b}{a^2b^2} = a^2b^{-1}$

$$u_n = u_1 \times r^{n-1} = a^2b^2 \times \left(a^2b^{-1}\right)^{n-1} = a^2b^2 \times (a^{2n-2}b^{1-n})$$
$$u_n = a^{2n}b^{3-n}$$

17 Let u_n be the cost of the nth metre in thousand dollars.

$u_1 = 10$ and $\{u\}$ follows an arithmetic progression with common difference $d = 0.5$.

$$S_n = \frac{n}{2}\big(2u_1 + d(n-1)\big)$$
$$S_{200} = 100(20 + 0.5 \times 199) = 11\,950$$

A 200 m tunnel will cost \$11.95 million.

18 Let u_n be the amount deposited on Elsa's $(n-1)$th birthday, in dollars.

$u_1 = 100$ and $\{u\}$ follows an arithmetic progression with common difference $d = 50$

$$S_n = \frac{n}{2}\big(2u_1 + d(n-1)\big)$$
$$S_{19} = 9.5(200 + 50 \times 18) = \$10\,450$$

Tip: Note that this working used S_{19} because the standard formula for S_n gives $\sum_{r=1}^{n} u_r$. This requires defining u_n so that the first value is given as u_1. The 18th birthday gift was therefore u_{19}.

An alternative approach would be to define v_n as the amount deposited on the nth birthday, so $v_1 = 150$.

Then the total up to and including the 18th birthday gift would be

$$100 + \sum_{r=1}^{18} v_r = 100 + \frac{18}{2}(2 \times 150 + 50 \times 17) = 100 + 1\,350 = 1\,450$$

The answer is of course the same, and the method is entirely for the mathematician to choose; just make sure you define the terms in your sequence clearly, for both your own and your readers' benefit.

19 The annual real terms change multiplier is $1 + r\% \approx \frac{1+c\%}{1+i\%} = \frac{1.058}{1.0292} = 1.028$

So over 3 years, the real terms percentage increase is $1.028^3 - 1 = 8.63\%$

20 The annual real terms change multiplier is $1 + r\% \approx \frac{1+c\%}{1+i\%} = \frac{0.9}{1.02} = 0.882$

So after 4 years, the real terms value is $\$2000 \times 0.883^4 = \$1212.27$

21 Under scheme A, his balance after n years is $\$(1000 + 25n)$

Under scheme B, his balance after n years is $\$(1000 \times 1.02^n)$

By calculator, $A > B$ for $n < 22.7$

So, for the first 22 years, A is better than B.

22 a $a_1 = 6$

b i $a_2 = 8$

ii $a_3 = 10$

c $d = 2$

d i $a_n = 4 + 2n$

If the final (Nth) pumpkin has $a_N = 48$ (distance to the pumpkin and back to the start line) then $N = 22$.

ii $S_n = \frac{n}{2}\left(2a_1 + d(n-1)\right)$

$S_{22} = 11(12 + 2 \times 21) = 594$

Sirma ran a total of 594 m.

e $S_m = \frac{m}{2}\left(12 + 2(m-1)\right) = 940$

$m^2 + 5m = 940$

By calculator, $m = 28.3$

So Peter has completed 28 runs and has collected 28 pumpkins.

f $S_{28} = 14(12 + 2 \times 27) = 924$

The 29th pumpkin is $a_{29} = 62$ m away from the start line.

Peter has run $940 - 924 = 16$ m of the way through the 29th collection, so he has not reached the pumpkin and is still on the outward journey, 16 m away from the start line.

23 Let the arithmetic sequence be $\{u\}$ with common difference d and the geometric sequence be $\{v\}$ with common ratio r.

Then $u_n = a + d(n-1)$ and $v_n = v_1 \times r^{n-1}$

$u_7 = v_1 = a + 6d$ (1)

$u_3 = v_2 = a + 2d = v_1 r$ (2)

$u_1 = v_3 = a = v_1 r^2$ (3)

$(2) \div (1): r = \frac{a+2d}{a+6d}$ (4)

$(3) \div (2): r = \frac{a}{a+2d}$ (5)

Equating (4) and (5): $\frac{a+2d}{a+6d} = \frac{a}{a+2d}$

$$(a+2d)^2 = a(a+6d)$$
$$a^2 + 4ad + 4d^2 = a^2 + 6ad$$
$$2ad = 4d^2$$

Then $a = 2d$ as required.

b $v_1 = u_7 = 3 = a + 6d = 4a$ so $a = \frac{3}{4}$ and $d = \frac{3}{8}$

Then $r = \frac{a}{a+2d} = \frac{1}{2}$

$$\sum_{r=1}^{n} u_r = \frac{n}{2}\big(2a + d(n-1)\big) = \frac{n}{2}\left(\frac{3}{4} + \frac{3}{8}(n-1)\right) = \frac{6n^2 + 9n}{16}$$

$$\sum_{r=1}^{n} v_r = v_1 \frac{(1 - r^n)}{1 - r} = 6\left(1 - \frac{1}{2^n}\right)$$

Require the least n such that $\frac{6n^2+9n}{16} \geqslant 200 + 6\left(1 - \frac{1}{2^n}\right)$

From the calculator, $n \geq 31.7$ so the least such n is 32.

24 a Under program A, she runs $a_m = 10 + 2(m-1)$ km on day m.

Under program B, she runs $b_n = 10 \times 1.15^{n-1}$ km on day n.

If $a_m \geqslant 42$ then $m \geqslant 17$; she first reaches 42 km on day 17.

If $b_n \geqslant 42$ then $n \geqslant 11.3$; she first reaches 42 km on day 12.

b $\{a\}$ follows an arithmetic progression with $a_1 = 10$ and $d = 2$

$$\sum_{r=1}^{m} a_r = \frac{m}{2}\big(2a_1 + d(m-1)\big) = \frac{m}{2}\big(20 + 2(m-1)\big) = m^2 + 9m$$

$m^2 + 9m \geqslant 90$ for $m \geqslant 6$ so under program A she completes 90 km on day 6.

$\{b\}$ follows a geometric progression with $b_1 = 10$ and $r = 1.15$

$$\sum_{r=1}^{n} b_r = b_1 \frac{r^n - 1}{r - 1} = 10\,\frac{1.15^n - 1}{0.15}$$

$10\,\frac{1.15^n-1}{0.15} \geq 90$ for $m \geq 6.1$ so under program B she completes 90 km on day 7.

25 a Let u_n be the salary in year n, in thousand pounds.

$\{u\}$ follows an arithmetic progression with $u_1 = 25$ and common difference $d = 1.5$

$u_n = u_1 + d(n-1)$ so $u_{30} = 25 + 1.5 \times 29 = 68.5$

Final year salary is £68 500

b $S_n = \frac{n}{2}\big(2u_1 + d(n-1)\big)$

$S_{30} = 15(50 + 1.5 \times 29) = £1\,402\,500$

c $\frac{68\,500}{1.015^{30}} = 43\,800$

In terms of the value at the beginning of the 30 year career, the final year salary has real value £43 800.

26 $u_{10} = u_1 + 9d$

$u_4 = u_1 + 3d$

If $u_{10} = 2u_4$ then $u_1 + 9d = 2u_1 + 6d$

Rearranging: $u_1 = 3d$ so $\frac{u_1}{d} = 3$

27 If the common difference of the sequence is k then

$$b = a + k$$

$$c = a + 2k$$

$$d = a + 3k$$

Then

$$2(b - c)^2 = 2(-k)^2 = 2k^2$$

$$\begin{aligned} bc - ad &= (a + k)(a + 2k) - a(a + 3k) \\ &= a^2 + 3ak + 2k^2 - (a^2 + 3ak) \\ &= 2k^2 \end{aligned}$$

Putting these together: $2(b - c)^2 = bc - ad$ as required.

3 Core: Functions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 3A

26 a $g(-2) = 4(-2) - 5$

$= -13$

b $4x - 5 = 7$ so $x = 3$

27

$$\frac{x-5}{3} = 12$$
$$x - 5 = 36$$
$$x = 41$$

28 a $v(1.5) = 3.8(1.5)$

$= 5.7 \text{ m s}^{-1}$

b It is unlikely that a car can accelerate uniformly at that level for 30 seconds (resistive forces that relate to speed would become significant before that time); the model would predict a speed of 114 m s^{-1} at 30 seconds, which is unrealistic for a car (over 400 km per hour!)

29 a Require that the denominator not equal zero: The largest domain of $f(x)$ is $x \neq 5$

b

$$f(2) = \frac{3}{(2-5)^2}$$
$$= \frac{3}{3^2}$$
$$= \frac{1}{3}$$

30 a

$$q\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2$$
$$= \frac{3}{4} - 2$$
$$= -\frac{5}{4}$$

b The function is in completed square form, so has minimum value -2.

The range is $q(x) \geqslant -2$

c $3x^2 - 2 = 46$

$3x^2 = 48$

$x^2 = 16$

$x = \pm 4$

31 a $N(5) = 2.3e^{0.49} + 1.2 = 4.95$

The model predicts 4.95 billion smartphones in 5 years

b Market saturation (population that can afford a smartphone having sufficient, not buying additional without discarding/recycling one) might slow the exponential growth. Smartphones may get replaced by a newer technology within a 5 year period.

32 a $f(x) = 1.3x$

b $f^{-1}(x)$ would return the amount in pounds equal to $\$x$

33 a $f^{-1}(6) = 1$

b $f(x) = x + 2$ has solution $x = 2$

34 a Require that the square root has a non-negative argument.

$2x - 5 \geqslant 0$ so the largest possible domain is $x \geqslant \frac{5}{2} = 2.5$

b If the argument of the square root takes any value greater than or equal to zero then the range of the function is $f(x) \geqslant 0$.

c $\sqrt{2x - 5} = 3$

$2x - 5 = 9$

$2x = 14$

$x = 7$

35 a At $t = 0$, $N = 150 - 90 \times 1 = 60$ fish

b

$N(15) = 150 - 90e^{-1.5}$

$= 130$ fish (rounding to the nearest whole number)

c The model is continuous (it predicts non-integer values). Such a model is an acceptable approximation for populations numbering in the millions (such as microbial populations in a culture) but the inaccuracy becomes more relevant for a small population such as seen here.

36 a $f(18) = \frac{18}{2} + 5$

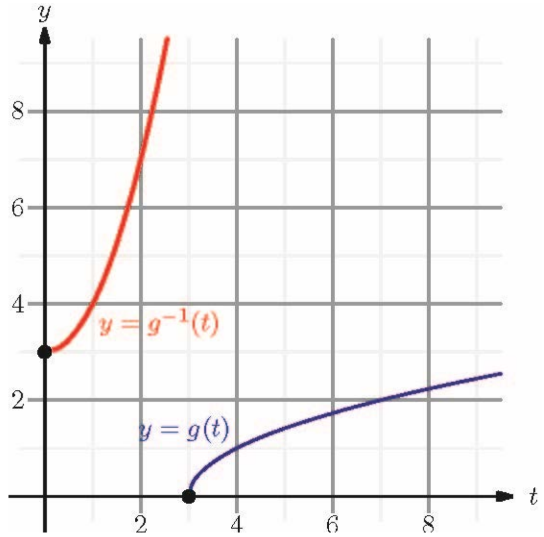
$= 14$

b $\frac{x}{2} + 5 = 7$

$\frac{x}{2} = 2$

$x = 4$

37 a The graph of the inverse function is the original graph after a reflection through $y = x$.



38 a Require that the logarithm takes a positive argument so the largest possible domain of g is $x > 0$.

b

$$\begin{aligned} g(81) &= \log_3 81 \\ &= \log_3 3^4 \\ &= 4 \end{aligned}$$

c $\log_3 x = -2$

$$\begin{aligned} x &= 3^{-2} \\ &= \frac{1}{9} \end{aligned}$$

39 Require that the logarithm has a positive argument.

$3x - 15 > 0$ so the largest domain of $n(x)$ is $x > 5$

40 a Require that the logarithm has a positive argument.

$7 - 3x > 0$ so the largest domain of $h(x)$ is $x < \frac{7}{3}$

b $\log(7 - 3x) = 2$

$$\begin{aligned} 7 - 3x &= 10^2 \\ 3x &= 7 - 100 \\ x &= -31 \end{aligned}$$

41 a $f(-3) = 10 - 3(-3)$

$= 19$

b The domain is $x \leqslant 2$ so the range is $f(x) \geqslant 10 - 3(2)$

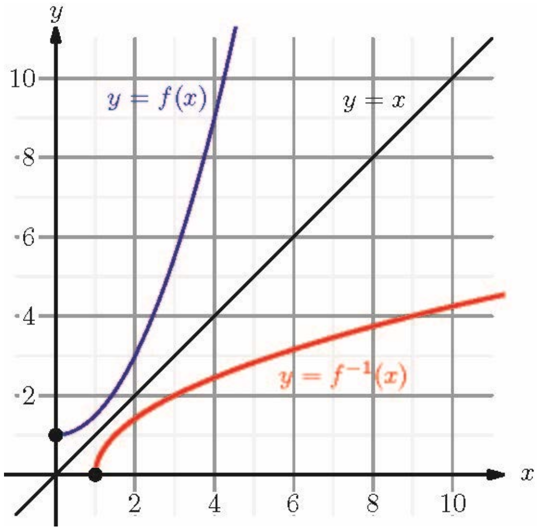
$f(x) \geqslant 4$

c The value 1 lies outside the range of the function.

42 a $f(4) = 9$

b $f^{-1}(4) \approx 2.5$

c The graph of the inverse function is the original graph after a reflection through $y = x$.



43 a

$$N(7) = 3e^{-2.8}$$
$$= 0.182$$

b $3e^{-0.4t} = 2.1$

$$t = -\frac{1}{0.4}\ln 0.7 = 0.892$$

44 a Require that the denominator is non-zero, so $4 - \sqrt{x-1} \neq 0$

$x - 1 \neq 16$

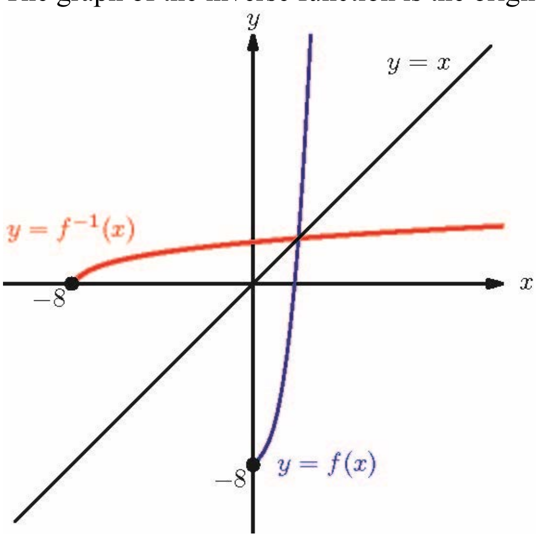
Also require that the argument of the square root is non-negative so $x - 1 \geqslant 0$

Then the largest domain is $x \geqslant 1, x \neq 17$

b From calculator (or by considering the two cases $4 - \sqrt{x-1} > 0$ and $4 - \sqrt{x-1} < 0$), the range is $f(x) \geqslant \frac{3}{4}$ or $f(x) < 0$

45 a Using GDC

b The graph of the inverse function is the original graph after a reflection through $y = x$.



c Either from GDC or by noting that where $f(x) = f^{-1}(x)$ the curves meet the line $y = x$ as well.

Thus $f(x) = x$ at the solution.

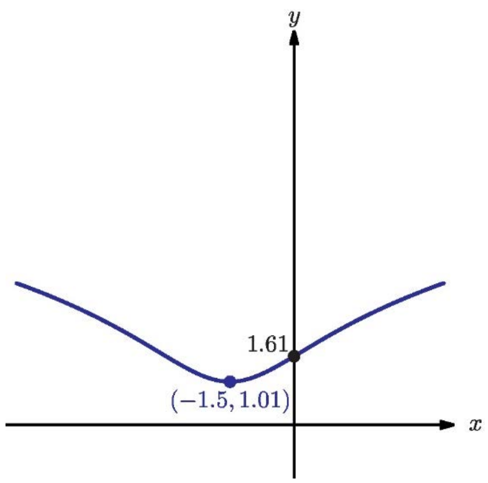
Then $x^3 + x - 8 = x$ so $x^3 = 8$, which gives solution $x = 2$.

Tip: It is tempting to hope that this always works to solve $f(x) = f^{-1}(x)$ but this is not the case for every function; you should always consider the graphs of f and f^{-1}. Consider the function $f(x) = 1 - x$. Since $f^{-1}(x) = 1 - x$ as well (the function is "self-inverse"), you can see that $f(x) = f^{-1}(x)$ for all values of x, even though the only solution on the line $y = x$ is $x = \frac{1}{2}$.

See if you can describe a condition on the graph of a function $h(x)$ for there to be solutions to $h(x) = h^{-1}(x)$ which do not lie on $y = x$.

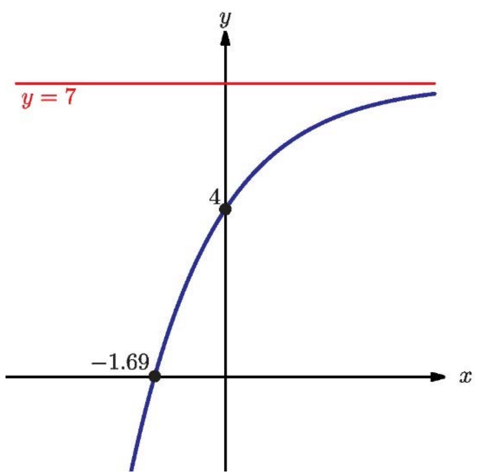
Exercise 3B

19



From GDC: Vertex is at $(-1.5, 1.01)$

20

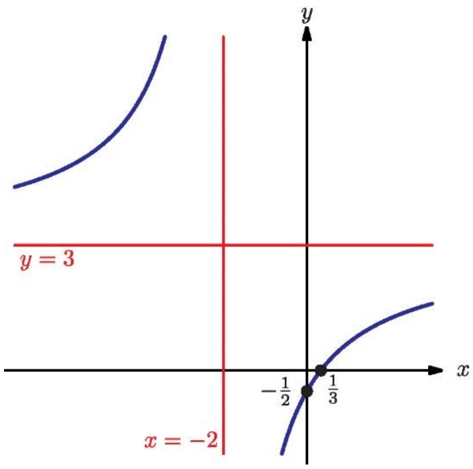


Asymptote $y = 7$; axis intercepts $(-1.69, 0)$ and $(0, 4)$

21 **a** Require denominator to be non-zero.

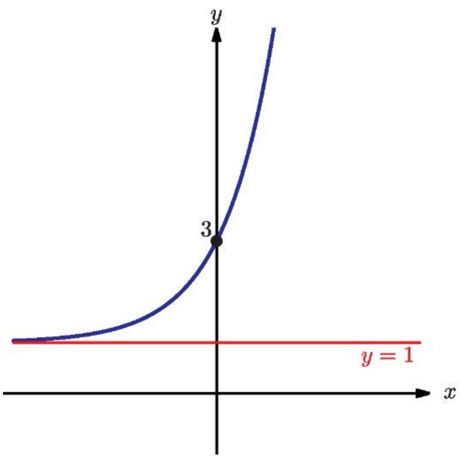
Largest domain is $x \neq 2$

b



Asymptotes are $y = 3$ and $x = -2$

22 For example, $y = 1 + 2\mathrm{e}^x$



23 From GDC, the intersection of $y = 5 - x$ and $y = \frac{1}{2}\mathrm{e}^x$ is $(1.84, 3.16)$

24 From GDC, maximum point is $(-1, 2.5)$

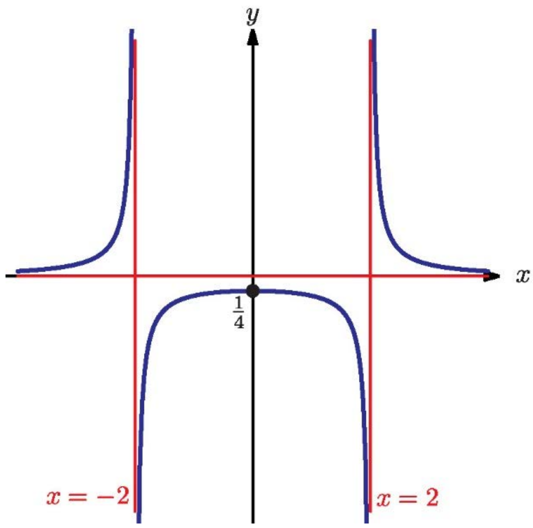
(Analytically, complete the square of the denominator and reason that since the denominator is always positive, the maximum will occur at the minimum value of the denominator, at $x = -1$.

$$y = \frac{10}{(x+1)^2 + 4}$$

25 From GDC, maximum value P occurs when $q = 235$

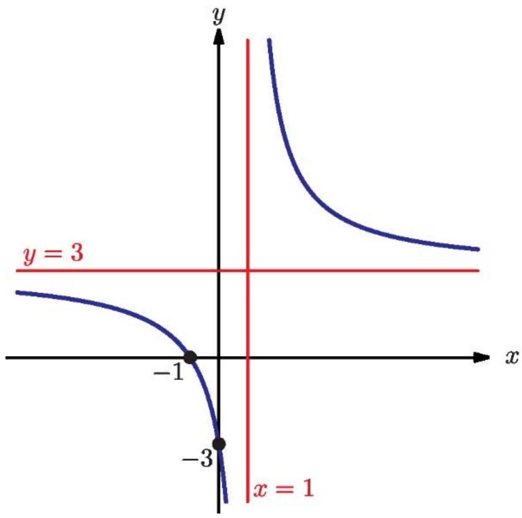
26 From GDC or completing the square to $y = \frac{1}{(x+1)^2+2}$, the line of symmetry is $x = -1$

27



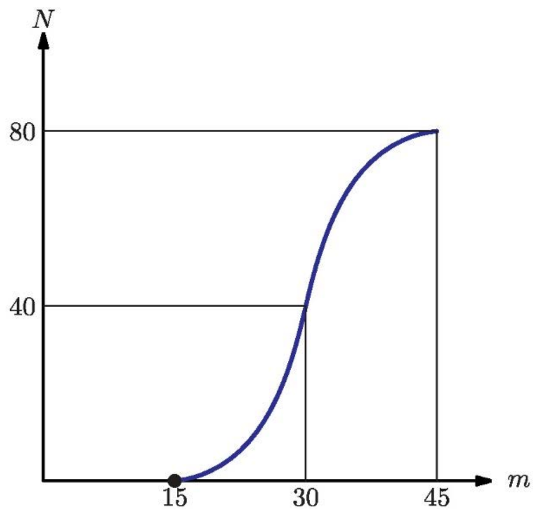
Vertical asymptotes are $x = -2$ and $x = 2$, horizontal asymptote is $y = 0$.

28

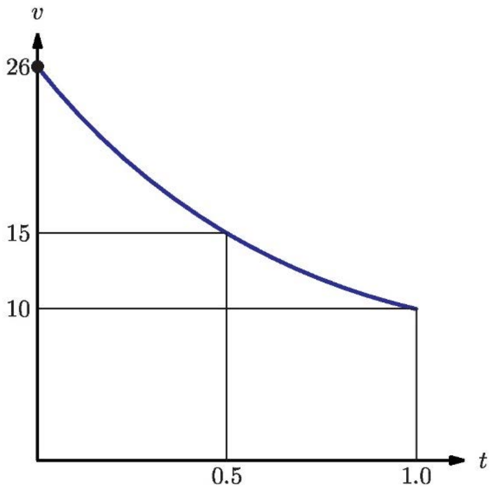


(Example: $y = \frac{3x+3}{x-1}$)

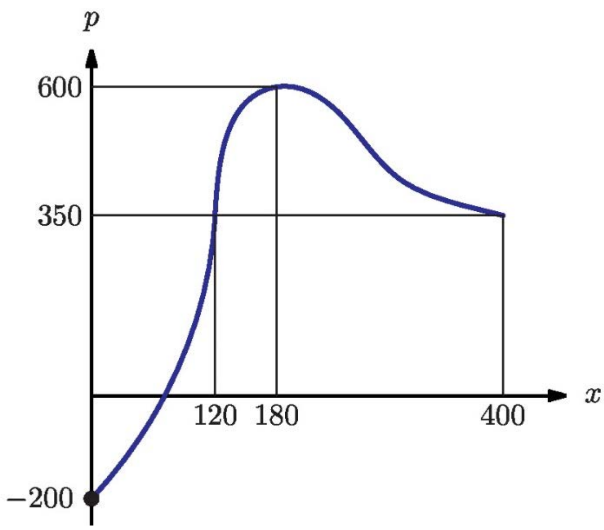
29



30



31



32 From GDC: $x = 0.755$

33 From GDC: $x = -2.20, -1.71$ or 1.91

34 From GDC: $x = -2.41, 0.414, 2$

35 From GDC: $x = \pm 1.41$

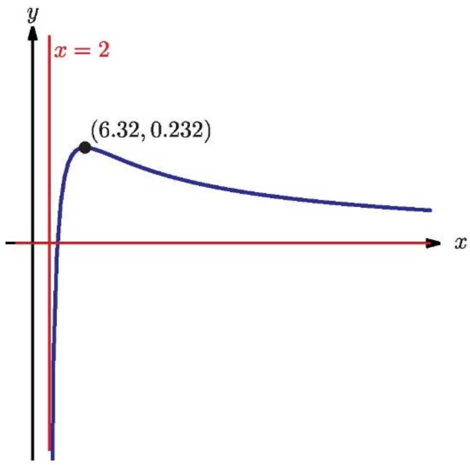
Solving exactly: $|4 - x^2| = x^2$

$$4 - x^2 = \pm x^2$$
$$4 = 0 \text{ (reject) or } 4 = 2x^2$$
$$x = \pm\sqrt{2} = \pm 1.41$$

36 From the calculator, maximum value is $f(2.5) = 0.920$

Tip: If you study calculus, you will learn to show algebraically that the maximum value occurs at $x = 2.5$)

37



38 From GDC: $x \geqslant 2$

Algebraically: $\ln|x-1| = |\ln(x-1)|$.

LHS equals a modulus function so must be non-negative. $\ln|x-1| \geqslant 0$ so $x \geqslant 2$

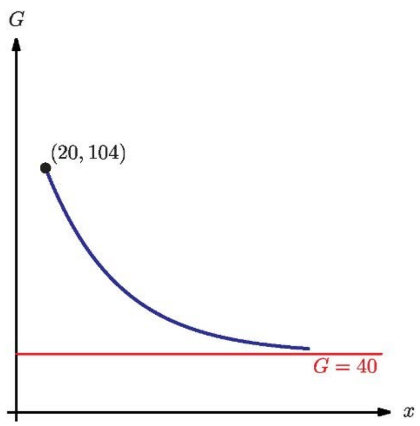
But for $x \geqslant 2, \ln|x-1| = \ln(x-1)$ and $\ln(x-1) \geqslant 0$ so $\ln|x-1| = \ln(x-1)$

So for $x \geqslant 2$, the two sides are always equivalent, with no further restriction.

Therefore the full solution is just $x \geqslant 2$.

Mixed Practice

1 **a**



b $G(45) = 78.6$

So the total cost is $45 \times G(45) = \$3\,538.09 \approx \$3\,540$

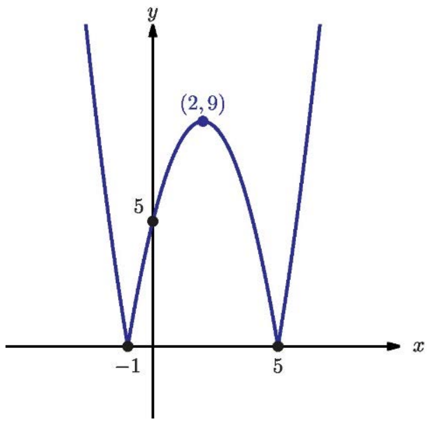
2 From GDC: $x = 1.86$ or 4.54

3 **a** Require that the argument of the square root is non-negative so require $x \geqslant -5$

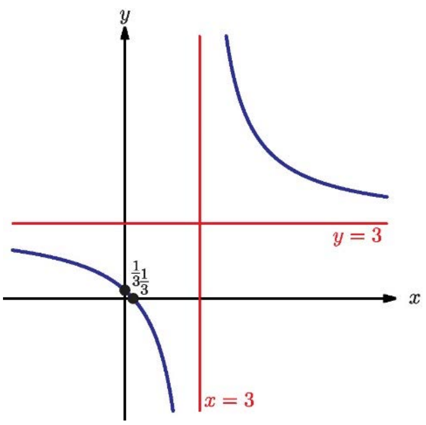
b From GDC, $x = -2.38$

4 From GDC, intersection points are $(-1.68, 0.399)$ and $(0.361, 2.87)$

5



6 **a**



b Domain: $x \neq 3$

Range: $f(x) \neq 3$

7 **a** $v(1.5) = 18 \times 1.5e^{-0.3} \approx 20.0 \text{ m s}^{-1}$

b From GDC: At $t = 0.630$ s and $t = 17.1$ s

c Maximum speed occurs at $t = 5$ s

8 **a** $v_1(0) = 8 - 6 = 2$

$v_2(0) = 2 + 0 - 0 = 2$

$v_1(0) = v_2(0)$ so the two runners have the same initial speed.

b From GDC, $v_1(t) = v_2(t)$ when $t = 1.30$ s

9 **a** $T(0) = 100$. Boiling point at sea level is 100 °C

b $T(1370) = 100 - 0.0034 \times 1370 = 95.3$ °C

c $70 = 100 - 0.0034h$ so $h = \frac{30}{0.0034} = 8\,820 \text{ m} = 8.82 \text{ km}$

10 **a** $\text{f}(x) = 1.8x + 32$

b $\text{f}^{-1}(x)$ gives the temperature in Celsius equivalent to x °F

11 **a** $p = \text{f}(1) = 3 - 5 = -2$

$3q - 5 = 7$ so $q = 4$

b i Require that the denominator is non-zero, so the denominator is $x \neq 2$

ii The horizontal asymptote is $y = 0$ so the range of the function is $g(x) > 0$

iii The vertical asymptote is $x = 2$

12 a $f(7) = 20$ and the function increases so the range is $f(x) \geqslant 20$

b $3x - 1 = 35$

$3x = 36$

$x = 12$

13 From GDC: $x = -5.24$ or 3.24

Algebraically:

$$2x - 1 = (4 - x)(4 + x)$$
$$= 16 - x^2 \text{ and } x \neq -4$$

$x^2 + 2x - 17 = 0$ and $x \neq -4$

$$x = -1 \pm 3\sqrt{2}$$

14 From GDC: $x = 1$ or 2.41

Algebraically:

$$x - 2 = \pm\frac{1}{x} \text{ and } x > 0 \text{ (since } \frac{1}{x} > 0)$$

$x^2 - 2x \pm 1 = 0$ and $x > 0$

$$(x - 1)^2 = 0 \text{ or } \left(x - 1 + \sqrt{2}\right)\left(x - 1 - \sqrt{2}\right) = 0 \text{ and } x > 0$$

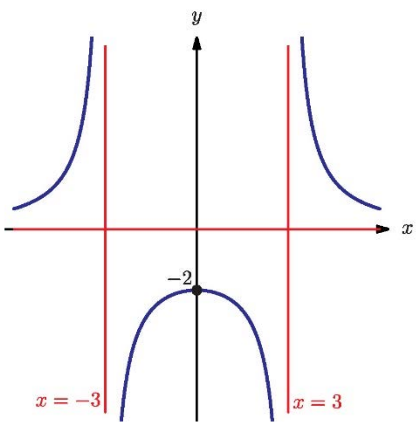
$x = 1$ or $x = 1 + \sqrt{2}$

15 Require denominator is non-zero so domain is $x \neq \pm 3$

From GDC (or symmetry), maximum point is at $x = 0$ so local max in the $-2 < x < 2$ interval is $h(0) = -2$.

For $x < -2$ and $x > 2$, $h(x) > 0$ with an asymptote of $y = 0$.

Hence the range is $h(x) \leqslant -2$ or $h(x) > 0$



16 From GDC, function has single stationary point at minimum $f(3.5) = -9.25$

The (unincluded) endpoints of the curve are $(0,3)$ and $(6,-3)$

The range is therefore $-9.25 \leqslant \mathrm{f}(x) < 3$

17 **a** From GDC, the minimum is $\mathrm{f}(\ln 0.8) = 4.89$

b $\mathrm{f}^{-1}(x) = 2$ so $x = \mathrm{f}(2) = 5\mathrm{e}^2 - 8 \approx 28.9$

18 **a** Function has maximum at $\mathrm{f}(0) = 8$.

The (unincluded) endpoints of the curve are $(-3,-19)$ and $(2,-4)$

The range is therefore $-19 < \mathrm{f}(x) \leqslant 8$

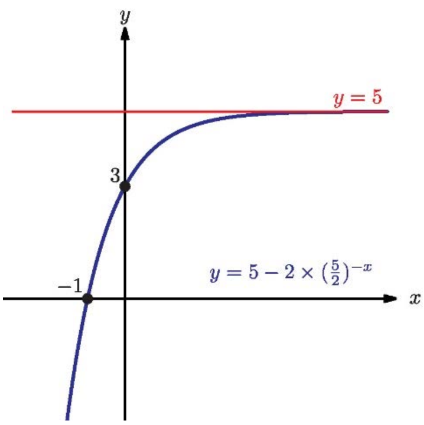
b $\mathrm{g}(x) = 8 - 3x^2 = 5$

$3x^2 = 3$

$x = 1, x = -1$

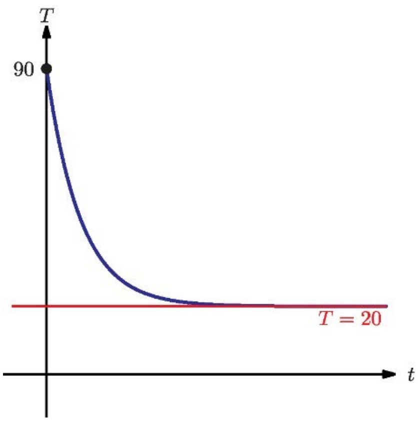
c -20 is outside the defined range of $\mathrm{g}(x)$.

19



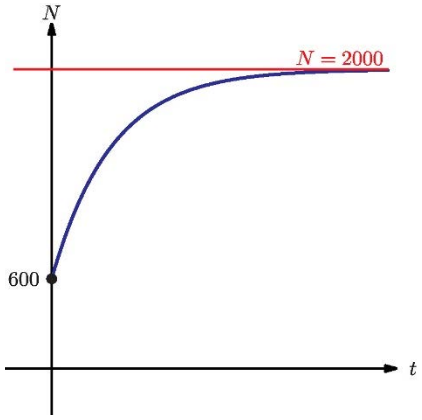
(Example: $y = 5 - 2 \times 0.4^x$)

20



(Example: $T = 20 + 70\mathrm{e}^{-t}$)

21



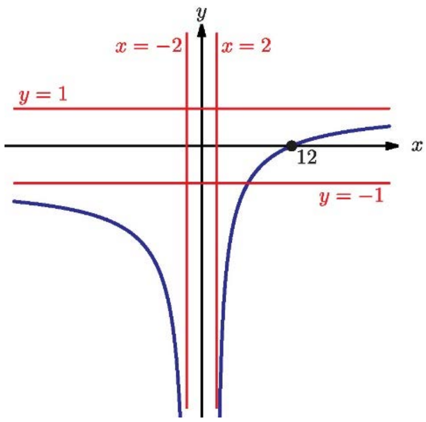
(Example: $N = 2000 - 1400e^{-t}$)

22 a From GDC, maximum is $v(1) = 1.10$ m s^{-1}

b The model predicts an insignificant speed at $t = 20$. Realistically in a physical system, the car would have halted before that time, when resistances become significant when compared to the model's predicted behaviour.

23 From the GDC, solutions are $x = -2.50, -1.51, 0.440$

24 a



b i x-intercept is at $(12,0)$

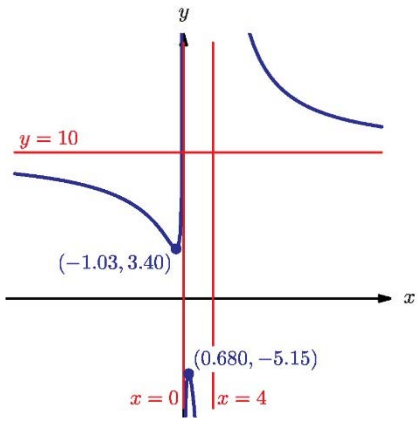
ii Asymptotes are $x = \pm 2$, $y = \pm 1$

25 From GDC, solutions are $x = 2.27$ or 4.47

26 From GDC, the minimum value is $f(1.16) = -1.34$

27 a Require the denominator is non-zero so the domain is $x \neq 0,4$

b



Range is $f(x) \leqslant -5.15$ or $f(x) \geqslant 3.40$

28 a Require denominator to be non-zero, so $x \neq e^{-3}$

Also require argument of logarithm to be positive, so $x > 0$

So the largest possible domain of $g(x)$ is $x > 0, x \neq e^{-3}$

b From GDC:

As $x \to 0$, $g(x) \to 0$ (from below)

As $x \to e^{-3}$ from below, $g(x) \to -\infty$

As $x \to e^{-3}$ from above, $g(x) \to \infty$

As $x \to \infty$, $g(x) \to \infty$

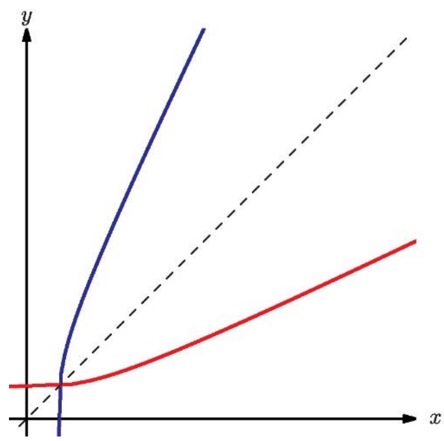
The curve has a single minimum at $f(0.135) = 0.271$

Tip: If you study further differentiation, see if you can show that the minimum has exact value $f(e^{-2}) = 2e^{-2}$. For this exercise though, the decimal approximation from your calculator is the faster approach, and is what the question requires.

29 a Require the argument of the logarithm to be positive, so the domain is $x > 2$

For $x > 2$, $g(x)$ has range $\mathbb{R}$.

b



c From GDC, $g(x) = g^{-1}(x)$ at $x = 2.12$

4 Core: Coordinate geometry

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 4A

34 a

$$\text{Gradient } m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-(-7)}{5-1} = \frac{7}{4}$$

b Perpendicular gradient $m = -\frac{1}{m_{AB}} = -\frac{4}{7}$

$$y - y_C = m(x - x_C)$$

$$y - 3 = -\frac{4}{7}(x - 8)$$

$$y = -\frac{4}{7}x + \frac{53}{7}$$

35 a Using the axis intercepts $(0,12)$ and $(9,0)$ to determine gradient:

$$\text{Gradient } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-12}{9-0} = -\frac{4}{3}$$

b $$y - y_1 = m(x - x_1)$$

$$y - 12 = -\frac{4}{3}(x - 0)$$

$$4x + 3y = 36$$

36 a Line $y = -\frac{5}{7}x + \frac{17}{7}$ has gradient $-\frac{5}{7}$

b When $y = 0$. $x = \frac{17}{5}$

37 a

$$\text{Gradient } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{7-(-3)} = \frac{1}{5}$$

b $y - y_1 = m(x - x_1)$

$$y - 1 = \frac{1}{5}(x - (-3))$$

$$5y - 5 = x + 3$$

$$x - 5y = -8$$

38 a Rearranging: $l_1: y = 21 - \frac{7}{2}x$

Gradient $m_1 = -\frac{7}{2}$

b Substituting $x = 8, y = -5$:

$7x + 2y = 56 - 4 = 52 \neq 42$

P does not lie on the line.

c Gradient $m_2 = -\frac{1}{m_1} = \frac{2}{7}$

$$y - y_1 = m(x - x_1)$$

$$y - (-5) = \frac{2}{7}(x - 8)$$

$$y = \frac{2}{7}x - \frac{51}{7}$$

39 Rise $= 8\,800$

Tread $= 12\,000$ (half the cone diameter)

Gradient $m = \frac{\text{Rise}}{\text{Tread}} = \frac{8800}{12000} = 0.733$

40 a $Q: (8,11)$

b $k = 8$

c QR is the vertical line $x = 8$

d Area $= \frac{1}{2}bh = \frac{1}{2} \times 8 \times 14 = 56$

41 a $y = -\frac{3}{2}x - 6$

b Gradient $m_2 = \frac{-2-1}{7-5} = -\frac{1}{4}$

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = -\frac{1}{4}(x - 7)$$

$$4y + 8 = -x + 7$$

$$x + 4y = -1$$

c From GDC, the lines intersect at $(-4.6, 0.9)$

42 If the pivot point is the origin, then the end of the plank is at (x, y) where $\frac{y}{x} = 0.7$ and $y = 3.5$

So $x = \frac{y}{0.7} = 5$

Then the plank length is $\sqrt{5^2 + 3.5^2} = 6.10$ m

43 a $V = 0.1 + 0.5t$

b When $V = 5, 0.5t = 4.9$

$t = 9.8$

It would take 9.8 seconds to pop the balloon.

44 a Newtons per metre (N/m)

b $F = 0.3x$

If $x = 0.06$ then $F = 0.018$

Require 0.018 N

c A stiffer spring will require more force for the same stretch; k will be greater.

d $x = \frac{F}{k}$

So when $F = 0.14$ and $k = 0.3$, the stretch is $\frac{0.14}{0.3} = 0.467$ m $= 46.7$ cm

45 a $C_1 = 5 + 0.01m$

b $C_1(180) = 6.8$

The cost is \$6.80 per month

c New model: $C_2 = 0.02m$

Intersection: $0.02m = 5 + 0.01m$

$m = 500$

The first contract will be better if Joanna expects to talk more than 500 minutes per month (6 hours 20 minutes).

46 a For n items sold in a single month, the monthly profits are

$P_1 = 10n - 2000$

b $P = 1500 = 10n - 2000$

$n = 350$

c

$$\begin{aligned} P_2 &= 10n - (1200 + 2n) \\ &= 8n - 1200 \end{aligned}$$

d Intersection: $10n - 2000 = 8n - 1200$

$2n = 800$

$n = 400$

Beyond this point, the first model predicts greater profits (steeper slope/no marginal costs).

For $P = 1500$, which is lower than the intersection point of the two models, the company will need to sell fewer items under the second model.

47 a Gradient of AB is $m_{AB} = \frac{8-3}{3-(-4)} = \frac{5}{7}$

Gradient of CD is $m_{CD} = \frac{-11-(-1)}{-9-5} = \frac{-10}{-14} = \frac{5}{7}$

So lines AB and CD are parallel, since they have the same gradient

b However, AB and CD are different lengths, so the shape $ABCD$ is not a parallelogram.

48 Considering the path as the hypotenuse of a right-angled triangle, the rise must be 400 m and $\frac{\text{rise}}{\text{tread}} = 0.3$

So tread $= \frac{\text{rise}}{0.3} = 1333$ m

Then the length of the path is $\sqrt{400^2 + 1333^2} = 1390$ m

Exercise 4B

10 a $\left(\frac{-4+7}{2}, \frac{1+0}{2}, \frac{9+2}{2}\right) = \left(\frac{3}{2}, \frac{1}{2}, \frac{11}{2}\right) = (1.5, 0.5, 5.5)$

b

$$AB = \sqrt{(7-(-4))^2 + (0-1)^2 + (2-9)^2}$$
$$= \sqrt{121 + 1 + 49}$$
$$= \sqrt{171}$$

11 $B: (b_x, b_y, b_z)$ where $(5, 1, -3) = \left(\frac{b_x+4}{2}, \frac{b_y-1}{2}, \frac{b_z+2}{2}\right)$

So B is $(6, 3, -8)$

12 $\left(\frac{-4+b}{2}, \frac{a+1}{2}, \frac{1+8}{2}\right) = (8, 2, c)$

$a = 3, b = 20, c = \frac{9}{2} = 4.5$

13 Midpoint M has coordinates $\left(\frac{3+2}{2}, \frac{-18-2}{2}, \frac{8+11}{2}\right) = (2.5, -10, 9.5)$

Distance $OM = \sqrt{2.5^2 + (-10)^2 + 9.5^2} = \sqrt{196.5}$

14 Distance travelled $d = \sqrt{14^2 + 3^2 + 6.7^2} = \sqrt{249.89}$

Average speed $= \frac{\text{total distance}}{\text{time taken}} = \frac{d}{3.5} = 4.52 \text{ m s}^{-1}$

15 $\sqrt{k^2(1^2+2^2+5^2)} = 30$

$k = \frac{30}{\sqrt{30}} = \sqrt{30}$ (selecting positive root for k)

16 $\sqrt{(k-1)^2+(k+1)^2+(-3k)^2} = \sqrt{46}$

$$11k^2 + 2 = 46$$
$$k^2 = 4$$
$$k = \pm 2$$

17 $\sqrt{a^2(2^2+1^2+5^2)} = 2\sqrt{(-4)^2+1^2+7^2}$

$$a = 2\frac{\sqrt{66}}{\sqrt{30}} = 2.97$$

18 a $\left(\frac{3a+1+5-b}{2}, \frac{2a+b+3}{2}\right) = (4,-5)$

$$3a - b + 6 = 8 \qquad (1)$$

$$2a + b + 3 = -10 \qquad (2)$$

$(1) + (2)$: $5a + 9 = -2$ so $a = -2.2$

(2): $b = -13 - 2a = -8.6$

b So $P: (-5.6, -4.4)$ and $Q: (13.6, -5.6)$

$$\text{Gradient } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5.6-(-4.4)}{13.6-(-5.6)} = \frac{-1.2}{19.2} = -\frac{1}{16}$$

$$y - y_1 = m(x - x_1)$$
$$y + 5 = -\frac{1}{16}(x - 4)$$

19 a Rearranging: $y = \frac{4}{7}x - 5$

Gradient $m_1 = \frac{4}{7}$

b Perpendicular gradient $m_2 = -\frac{7}{4}$

$$y - y_1 = m(x - x_1)$$
$$y - 2 = -\frac{7}{4}(x - (-4))$$
$$4y - 8 = -7x - 28$$
$$7x + 4y = -20$$

c $l_1: 4x - 7y = 35 \qquad (1)$

$l_2: 7x + 4y = -20 \qquad (2)$

$4(1) + 7(2)$: $65x = 0$ so $x = 0, y = -5$

$P: (0, -5)$

d Shortest distance will be perpendicular to l_1 and so equals NP.

$$NP = \sqrt{(0-(-4))^2 + (-5-2)^2}$$
$$= \sqrt{65}$$
$$= 8.06$$

20 a 1.8 m

b Midpoint of BC is $G: (3, 1, 0)$

$$GE = \sqrt{(3-0.2)^2 + (1-1)^2 + (0-1.8)^2}$$
$$= \sqrt{11.08}$$
$$= 3.33 \text{ m}$$

21 a Midpoint $M: (3,7)$

b From A to M: 2 units right and 3 units down

So from M to B will be the same translation, but rotated 90°: 2 units down and 3 units left

$B: (6,5)$

Then M is the midpoint of BD, since it is the centre of the square

$D: (0,9)$

22 a Gradient $m_{AC} = \frac{8-1}{8-1} = 1$

So Gradient $m_{BD} = -1$

The centre of the rhombus is M, the midpoint of AC

M has coordinates $(4.5, 4.5)$

So line BD has equation $y - 4.5 = -(x - 4.5)$

$y = 9 - x$ or $y + x = 9$

b $m_{AB} = \frac{4}{3}$ so line AB has equation $y - 1 = \frac{4}{3}(x - 1)$

$$y = \frac{4}{3}x - \frac{1}{3}$$

B is the intersection of these lines. Substituting:

$$9 - x = \frac{4}{3}x - \frac{1}{3}$$
$$\frac{7}{3}x = \frac{28}{3}$$

$x = 4$ so $y = 5$. B has coordinates $(4,5)$

Since M is the midpoint of BD, it follows that D has coordinates $(5,4)$

c $AB = \sqrt{(4-1)^2 + (5-1)^2} = 5$

23 a Diagonal of the cuboid has length $\sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \approx 7.07$ m

b Considering the net of the room, the shortest distance for the spider would be the diagonal of the rectangle formed by any two sides of the cuboid.

The options are:

Diagonal of a 3 by $(4+5)$: $\sqrt{3^2+9^2}=\sqrt{90}$

Diagonal of a 4 by $(3+5)$: $\sqrt{4^2+8^2}=\sqrt{80}$

Diagonal of a 5 by $(3+4)$: $\sqrt{5^2+7^2}=\sqrt{74}\approx 8.60$ m

Mixed Practice 4

1 **a** **i** $M: \left(\frac{4+0}{2}, \frac{1-5}{2}\right) = (2, -2)$

ii Gradient $m_{PQ} = \frac{-5-1}{0-4} = \frac{3}{2}$

iii Perpendicular gradient is $-\frac{2}{3}$

b Line has equation $y - y_1 = m(x - x_1)$

$$y + 2 = -\frac{2}{3}(x - 2)$$

Substituting $x = 0, y = k$: $k + 2 = \frac{4}{3}$ so $k = -\frac{2}{3}$

2 **a** $M: (2,3) = \left(\frac{s-2}{2}, \frac{8+t}{2}\right)$

So $s = 6, t = -2$

b Gradient of AB is $m_{AB} = \frac{t-8}{-2-s} = \frac{-10}{-8} = \frac{5}{4}$

So the perpendicular gradient is $m_{\perp} = -\frac{4}{5}$

Line has equation $y - y_1 = m(x - x_1)$

$$y - 3 = -\frac{4}{5}(x - 2)$$
$$5y - 15 = -4x + 8$$
$$4x + 5y = 23$$

3 **a** When $x = 6, y = 0$, $2y - 3x = 0 - 18 = -18 \neq 11$

Since $2y - 3x \neq 11$ at point A, it follows that A does not lie on L_1

b Rearranging: $y = \frac{3}{2}x + \frac{11}{2}$

Gradient of L_1 is $\frac{3}{2}$

c Perpendicular gradient $m_{\perp} = -\frac{2}{3}$

d Line has equation $y - y_1 = m(x - x_1)$

$$y - 0 = -\frac{2}{3}(x - 6)$$
$$y = -\frac{2}{3}x + 4$$
$$c = 4$$

4 **a** $m_{AB} = \frac{(10-6)}{-1-3} = -1$

b Perpendicular gradient $m_{\perp} = 1$

Line has equation $y - y_1 = m(x - x_1)$

$$y - 6 = 1(x - 3)$$
$$y = x + 3$$

c P has coordinates $(0,3)$ and Q has coordinates $(3,0)$

Area $OPQ = \frac{1}{2} \times 3 \times 3 = 4.5$

5 **a** $M: \left(\frac{-1+6}{2}, \frac{2-4}{2}\right) = (2.5, -1)$

b

$$PQ = \sqrt{\left(6-(-1)\right)^2 + (-4-2)^2}$$
$$= \sqrt{85}$$
$$= 9.22$$

c Gradient of PQ is $m_{PQ} = \frac{-4-2}{6-1} = -\frac{6}{7}$

Perpendicular gradient $m_{\perp} = \frac{7}{6}$

Line has equation $y - y_1 = m(x - x_1)$

$$y - (-1) = \frac{7}{6}(x - 2.5)$$
$$y = \frac{7}{6}x - \frac{47}{12}$$

6 **a** $M: \left(\frac{-1+6}{2}, \frac{2-4}{2}, \frac{5+3}{2}\right) = (2.5, -1, 4)$

b

$$PQ = \sqrt{\left(6-(-1)\right)^2 + (-4-2)^2 + (3-5)^2}$$
$$= \sqrt{89}$$
$$= 9.43$$

7 **a** When $x = 0$, $y = 3$ and when $y = 0$, $x = 6$ so P has coordinates $(0,3)$ and Q has coordinates $(6,0)$.

b $PQ = \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$

c Substituting $y = x$ into the equation for l_1: $3x = 6$ so $x = 2$, and the intersection is $(2,2)$

8 **a** Rearranging the line equation: $y = -\frac{7}{4}x + \frac{d}{4}$

Gradient $m_{MN} = -\frac{7}{4}$

b Gradient $m_{MN} = \frac{k-(-5)}{-1-3} = -\frac{k+5}{4}$

Equating with the answer to part **a**: $k = 2$

c Line through $N(-1,2)$ with gradient $m_{MN} = -\frac{7}{4}$ has equation

$$(y-2) = -\frac{7}{4}(x-(-1))$$
$$y = -\frac{7}{4}x + \frac{1}{4}$$

Equating with the answer to part **a**: $d = 1$

9 **a** $m_{AB} = \frac{5-8}{2-(-3)} = -\frac{3}{5}$

$$m_{DC} = \frac{(6-9)}{1-(-4)} = -\frac{3}{5}$$

So $AB \parallel DC$

$$m_{AD} = \frac{9-8}{-4-(-3)} = -1$$

$$m_{BC} = \frac{6-5}{1-2} = -1$$

So $AD \parallel BC$

The quadrilateral has two pairs of parallel sides, so is a parallelogram

b Since $m_{AD} \times m_{AB} \neq -1$, the angle at A is not a right-angle, so the shape is not a rectangle.

10 If the three vertices are $A(-2,5)$, $B(1,3)$ and $C(5,9)$

$$m_{AB} = \frac{3-5}{1-(-2)} = -\frac{2}{3}$$

$$m_{AC} = \frac{9-5}{5-(-2)} = \frac{4}{7}$$

$$m_{BC} = \frac{9-3}{5-1} = \frac{3}{2}$$

Then $m_{AB} \times M_{BC} = -1$ so $AB \perp BC$ and therefore the triangle has a right angle at B.

11 $\sqrt{(-4)^2 + a^2 + (3a)^2} = \sqrt{416}$

$$16 + 10a^2 = 416$$
$$a^2 = 400$$
$$a = \pm 20$$

12 **a** Midpoint M: $\left(\frac{2-6}{2}, \frac{p+5}{2}, \frac{8+q}{2}\right) = (-2,3,-5)$

$p = 1, q = -18$

b $A: (2,1,8), B: (-6,5,-18)$

$$AB = \sqrt{(2-(-6))^2 + (1-5)^2 + (8-(-18))^2}$$
$$= \sqrt{756}$$
$$= 27.5$$

13 $m = \frac{6}{4} = 1.5$

14 Taking the equations simultaneously: $\frac{1}{2}x - 3 = 2 - \frac{2}{3}x$

$$\frac{7}{6}x = 5$$

$x = \frac{30}{7}$ so $y = -\frac{6}{7}$

$$P: \left(\frac{30}{7}, -\frac{6}{7}\right)$$

Distance $OP = \frac{1}{7}\sqrt{30^2 + 6^2} = \frac{6}{7}\sqrt{26} = 4.37$

15 a Gradient of AB $m_{AB} = \frac{0-3}{5-(-4)} = -\frac{1}{3}$

The perpendicular gradient is $m_{\perp AB} = 3$ so line $y = 3x$ is perpendicular to AB.

b Gradient of AC $m_{AC} = \frac{7-3}{4-(-4)} = \frac{1}{2}$

The perpendicular gradient is $m_{\perp AC} = -2$

Midpoint of AC $M = (0,5)$

Line equation $y - y_1 = m(x - x_1)$

$$y - 5 = -2(x - 0)$$

l_2 has equation $y = -2x + 5$

c Intersection of $y = 3x$ and $y = -2x + 5$:

$$3x = -2x + 5$$

$$5x = 5$$

$x = 1$ so S has coordinates $(1,3)$

$$SA = \sqrt{\left(1 - (-4)\right)^2 + (3 - 3)^2} = 5$$

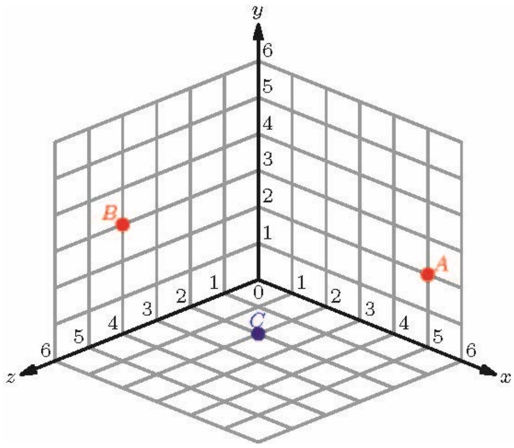
$$SB = \sqrt{(1 - 4)^2 + (3 - 7)^2} = 5$$

$$SC = \sqrt{(1 - 5)^2 + (3 - 0)^2} = 5$$

So point S is equidistant from the three points.

Tip: S is called the “circumcentre” of the triangle, and lies at the common intersection of all three side perpendicular bisectors. Because it is equidistant from all three vertices, a circle drawn with centre at the circumcentre through one of the vertices will also pass through the others as well, so that the triangle is inscribed exactly in a circle. You may like to investigate properties of the other major triangle “centre points” called orthocentre, centroid and incentre, and their relationships.

17 a



b $M: \left(\frac{5+0}{2}, \frac{2+3}{2}, \frac{0+4}{2}\right) = (2.5, 2.5, 2)$

c

$$AB = \sqrt{(0-5)^2 + (3-2)^2 + (4-0)^2}$$
$$= \sqrt{42}$$
$$= 6.48$$

18 a Gradient $m_{AB} = \frac{8-2}{5-(-7)} = \frac{1}{2}$

b $M: \left(\frac{5-7}{2}, \frac{2+8}{2}\right) = (-1,5)$

c Perpendicular gradient $m_1 = -2$

Equation of l_1:

$$y - y_1 = m(x - x_1)$$
$$y - 5 = -2(x - (-1))$$
$$2x + y = 3$$

d Substituting $x = 1, y = 1$ into the equation of l_1:

$2x + y = 2 + 1 = 3$ so N does lie on l_1

e Then the distance from N to line AB is the distance MN

$$MN = \sqrt{(1-(-1))^2 + (1-5)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5} = 4.47$$

19 a When $x = 0$, $y = -6$ so $A: (0,6)$ is the y-axis intercept

When $y = 0, x = 14$ so $B: (14,0)$ is the x-axis intercept

So triangle AOB has area $\frac{1}{2} \times 6 \times 14 = 42$

b

$$AB = \sqrt{6^2 + 14^2}$$
$$= \sqrt{232}$$
$$= 15.2$$

c Since the area can be calculated using any base side, and $AB = \sqrt{232}$, it follows that the distance from AB to the vertex O is $2 \times \frac{42}{\sqrt{232}} = 5.51$

20 When $y = 0$ on l_1 then $x = 10$ so $P: (10, 0)$

When $y = 0$ on l_2 then $x = \frac{9}{2}$ so $Q: \left(\frac{9}{2}, 0\right)$

Intersecting the lines: $l_1: x = 10 - 2y$ and $l_2: x = \frac{3}{2}y + \frac{9}{2}$

$$10 - 2y = \frac{3}{2}y + \frac{9}{2}$$
$$\frac{7}{2}y = \frac{11}{2}$$
$$y = \frac{11}{7}$$

So R has y-coordinate $\frac{11}{7}$ which is the altitude of triangle PQR , since the base lies along the x-axis.

Area $PQR = \frac{1}{2} \times \frac{11}{7} \times \frac{9}{2} = 4.32$

21 a Midpoint M of AC, which is the midpoint of the square, is $M: \left(\frac{8+2}{2}, \frac{3+1}{2}\right) = (5, 2)$

Gradient of AC is $m_{AC} = \frac{3-1}{2-8} = -\frac{1}{3}$

So the perpendicular gradient $m_{\perp AC} = m_{BD} = 3$

Then the equation of the other diagonal is $y - 2 = 3(x - 5)$

BD has equation $y = 3x - 13$

b From M to A is translation 3 right and 1 down, so from M to B will be a rotation 90° of this: 3 up and 1 right, so B is $(6, 5)$ and then since M is the midpoint of BD, D has coordinates $(4, -1)$

22 Midpoint of AC is $M_{AC} = \left(\frac{-3+9}{2}, \frac{2+(-2)}{2}\right) = (3, 0)$

Midpoint of $BD$is $M_{BD} = \left(\frac{4+2}{2}, \frac{3+(-3)}{2}\right) = (3, 0)$

So the two diagonals have a common midpoint.

Gradient of AC is $m_{AB} = \frac{-2-2}{9—3} = -\frac{1}{3}$

Gradient of BD is $m_{BD} = \frac{-3-3}{2-4} = 3$

$m_{AC} \times m_{BD} = -1$ so the two diagonals are perpendicular.

Since only a rhombus has bisecting diagonals which are perpendicular, it follows that $ABCD$ is a rhombus.

Tip: There are several alternatives here – pick a set of defining properties of a rhombus and show that they are true; an alternative would be to show that $AB \parallel CD$ and $AD \parallel BC$ (so that it is shown to be a parallelogram) and then also show that $AB = AD$.

As another option, ignore gradients altogether, and show that $AB = BC = CD = AD$

23 Gradient $0.15 = \frac{\text{Rise}}{\text{Tread}}$

So the horizontal distance is $\text{Tread} = \frac{\text{Rise}}{0.15} = \frac{20}{0.15} = 133.3$ m

Then the distance travelled (the hypotenuse of the triangle with vertical distance 20 and horizontal distance 133.3 is $\sqrt{20^2 + 133.3^2} = 135$ m

24 If the vertical distance of each section is 6 m and the gradient is 0.75 then the horizontal distance (when elevated) is $\frac{6}{0.75} = 8$ m.

The length of each section is therefore $\sqrt{6^2 + 8^2} = 10$ m

Then by elevating the bridge section, the ends each move 2 m from the midpoint of the bridge, for a total 4 m separation.

5 Core: Geometry and trigonometry

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 5A

16

$$\begin{aligned}\text{Surface Area} &= 4\pi r^2\\ &= 4\pi \times (7.5)^2 \text{ cm}^2\\ &= 225\pi \text{ cm}^2\\ &= 707 \text{ cm}^2\end{aligned}$$

17

$$\begin{aligned}\text{Curved Surface Area} &= 2\pi r^2\\ &= 2\pi \times (3.2)^2\end{aligned}$$

$$\begin{aligned}\text{Circle Face Area} &= \pi r^2\\ &= \pi \times (3.2)^2\end{aligned}$$

$$\begin{aligned}\text{Total Surface Area} &= 3\pi \times (3.2)^2\\ &= 96.5 \text{ cm}^2\end{aligned}$$

18

$$\begin{aligned}\text{Base Area} &= \pi r^2\\ &= \pi \times (1.6)^2\end{aligned}$$

$$\begin{aligned}\text{Volume} &= \frac{1}{3} \times \text{Base Area} \times \text{height}\\ &= \frac{1}{3} \times 2.56\pi \times 3.1\\ &= 8.31 \text{ m}^3\end{aligned}$$

19

$$\begin{aligned}\text{Volume} &= \frac{2}{3}\pi r^3\\ &= \frac{128}{3}\pi\\ &= 134 \text{ cm}^3\end{aligned}$$

20

$$\begin{aligned}\text{Volume} &= \frac{4}{3}\pi r^3\\ &= \frac{4}{3}\pi \times (9.15)^3\\ &= 3\,210 \text{ cm}^3\end{aligned}$$

21 Converting all measurements to cm.

Cylinder:

$$\text{Curved Surface Area} = 2\pi rl = 2 \times \pi \times 5 \times 100 = 1000\pi$$

$$\text{Base Surface Area} = \pi r^2 = \pi \times 5^2 = 25\pi$$

$$\text{Volume} = \pi r^2 l = \pi \times 5^2 \times 100 = 2500\pi$$

Cone:

$$\text{Curved Surface Area} = \pi r\sqrt{r^2 + h^2} = \pi \times 5 \times \sqrt{125} = 25\pi\sqrt{5}$$

$$\text{Volume} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 5^2 \times 10 = \frac{250}{3}\pi$$

$$\text{Total Surface Area} = \pi\left(1000 + 25 + 25\sqrt{5}\right) = 3400 \text{ cm}^2$$

$$\text{Total Volume} = \pi\left(2500 + \frac{250}{3}\right) = 8120 \text{ cm}^3$$

22 Cylinder height is 59 cm

$$\text{Cylinder Volume} = \pi r^2 h = \pi \times 14^2 \times 59 = 11\,564\pi$$

$$\text{Hemisphere Volume} = \frac{2}{3}\pi r^3 = \frac{5\,488}{3}\pi$$

$$\text{Total Volume} = \left(11\,564 + \frac{5\,488}{3}\right)\pi = 42\,100 \text{ cm}^3 = 4.21 \times 10^4 \text{ cm}^3$$

23 **a** $r = \sqrt{17^2 - 12^2} = 12.0$ cm

b

$$\begin{aligned}\text{Curved Surface Area} &= \pi r l\\ &= \pi \times 12.0 \times 17\\ &= 643\end{aligned}$$

$$\begin{aligned}\text{Base Surface Area} &= \pi r^2\\ &= 145\pi\\ &= 456\end{aligned}$$

$$\begin{aligned}\text{Total Surface Area} &= 643 + 456\\ &= 1\,100\ \text{cm}^2\end{aligned}$$

24 a $V = \pi r^2 h$

$503.7 = \pi r^2 \times 12$

$$r = \sqrt{\frac{503.7}{12\pi}}$$

$= 3.61$ cm

b $3 \times 12.3 = 36.9$ cm

Tip: Remember that a tapered solid has one third the volume of a prism the same height and base area, so a tapered solid with the same volume and base area would be three times the height.

25 a

$$\begin{aligned}V &= \frac{1}{3}\pi r^2 h\\ &= 192\ \text{cm}^3\end{aligned}$$

b $V_{sphere} = \frac{4}{3}\pi r^3 = 192$

$$r = \sqrt[3]{192 \times \frac{3}{4\pi}}$$

$= 3.58$ cm

26 a Height of cone $h_{\text{Cone}} = 35 - 23 = 12$ cm

Radius $r = 9$ cm

$$\begin{aligned}\text{Slant height } l &= \sqrt{9^2 + 12^2}\\ &= 15\ \text{cm}\end{aligned}$$

b Curved SA of cone $S_{\text{Cone}} = \pi \times 9 \times 15 = 135\pi\ \text{cm}^2$

Side SA of cylinder $S_{Cyl} = 2\pi \times 9 \times 23 = 414\pi\ \text{cm}^2$

Base area of cylinder $S_{\text{Base}} = \pi \times 9^2 = 81\pi\ \text{cm}^2$

Total SA $= 630\pi\ \text{cm}^2 = 1980\ \text{cm}^2$

27 a A side face is isosceles with base 12 cm and perpendicular distance $\sqrt{15^2 - 6^2} = 3\sqrt{21}$ cm

Side face area $= \frac{1}{2} \times 12 \times 3\sqrt{21} = 82.5\ \text{cm}^2$

b Base area $= 12 \times 12 = 144 \text{ cm}^2$

Total surface area $= 4 \times 18\sqrt{21} + 144 = 474 \text{ cm}^2$

c Diagonal of the base has length $12\sqrt{2}$ cm so half the diagonal is $6\sqrt{2}$ cm

Pyramid height $h = \sqrt{15^2 - (6\sqrt{2})^2} = \sqrt{153}$ cm

$$\begin{aligned} V &= \frac{1}{3} \times 144 \times \sqrt{153} \\ &= 594 \text{ cm}^3 \end{aligned}$$

28 For the complete cone:

$$V = \frac{1}{3}\pi \times 2^2 \times 6 = 8\pi \text{ cm}^3$$

After the hole is bored, the volume removed is $\frac{2}{3}\pi \times 1^3 = \frac{2}{3}\pi$

So the end volume is $\left(8 - \frac{2}{3}\right)\pi = 22 \text{ cm}^3$

Slant length $l = \sqrt{2^2 + 6^2} = \sqrt{40}$ cm

Cone curved SA $= \pi \times 2 \times \sqrt{40}$

Hemisphere curved SA $= 2\pi \times 1^2 = 2\pi \text{ cm}^2$

Base ring area $= \pi(2^2 - 1^2) = 3\pi \text{ cm}^2$

$$\begin{aligned} \text{Total SA} &= \left(5 + 2\sqrt{40}\right)\pi \text{ cm}^2 \\ &= 55.4 \text{ cm}^2 \end{aligned}$$

29

$$\begin{aligned} \text{Total volume} &= \frac{2}{3}\pi(8^3 + 10^3) \\ &= 3170 \text{ mm}^3 \end{aligned}$$

$$\begin{aligned} \text{Curved area} &= 2\pi(8^2 + 10^2) \\ &= 328\pi \text{ mm}^2 \end{aligned}$$

Ring area $= \pi(10^2 - 8^2) = 36\pi \text{ mm}^2$

$$\begin{aligned} \text{Total SA} &= 364\pi \text{ mm}^2 \\ &= 1140 \text{ mm}^2 \end{aligned}$$

30 Main cone:

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \times 8^2 \times 30 \\ &= 640\pi \text{ mm}^3 \end{aligned}$$

$$\begin{aligned} \text{Curved SA} &= \pi r l \\ &= \pi \times 8 \times \sqrt{8^2 + 30^2} \\ &= 248.4\pi \text{ mm}^2 \end{aligned}$$

Removed cone:

$$r = 8 \times \frac{12}{30} = 3.2 \text{ mm}$$

$$V = \frac{1}{3}\pi \times 3.2^2 \times 12$$
$$= 40.96\pi \text{ mm}^3$$

$$\text{Curved SA} = \pi \times 3.2 \times \sqrt{3.2^2 + 12^2}$$
$$= 39.7\pi \text{ mm}^2$$

$$\text{Frustum } V = 640\pi - 40.96\pi$$
$$= 1880 \text{ mm}^3$$

$$\text{Frustum SA} = (248.4\pi - 39.7\pi) + \pi(8^2 + 3.2^2)$$
$$= 889 \text{ mm}^2$$

31 **a** Let x be the length of one side of the base, so that the base area is x^2

$$V = \frac{1}{3}x^2h$$
$$x^2 = \frac{3V}{h} = \frac{3 \times 1352}{24} = 169$$
$$x = 13 \text{ cm}$$

b Let l be the altitude of one of the isosceles triangle faces.

$$l = \sqrt{h^2 + 0.25x^2}$$
$$= \sqrt{24^2 + 6.5^2}$$
$$= 24.86 \text{ cm}$$

Then the area of one of the triangles A is given by

$$A = \frac{1}{2}lx = 162 \text{ cm}^2$$

So the total surface area is $4A + x^2 = 815 \text{ cm}^2$

32 $V = \frac{4}{3}\pi r^3 = 354$

$$r = \sqrt[3]{\frac{3}{4} \times \frac{354}{\pi}} = 4.39$$

$$SA = 4\pi r^2 = \frac{3V}{r} = 242 \text{ m}^2$$

33 **a** Cylinder:

$$\text{Curved SA} = 2\pi rh$$
$$= 2\pi \times 2 \times 8$$
$$= 32\pi \text{ mm}^2$$
$$V = \pi r^2 h$$
$$= \pi \times 2^2 \times 8$$
$$= 32\pi \text{ mm}^2$$

Worked solutions

Hemispheres:

$$\begin{aligned}\text{Curved SA} &= 4\pi r^2\\ &= 4\pi \times 2^2\\ &= 16\pi \text{ mm}^2\\ V &= \frac{4}{3}\pi r^3\\ &= \frac{4}{3}\pi \times 2^3\\ &= \frac{32}{3}\pi \text{ mm}^3\end{aligned}$$

Total:

$$\text{SA} = 48\pi \text{ mm}^2 = 151 \text{ mm}^2$$

$$V = \frac{128}{3}\pi \text{ mm}^3 = 134 \text{ mm}^3$$

b Require $r = 1.8$ mm and $V = 0.9 \times 134 = 121 \text{ mm}^3$

$$V = \pi r^2\left(h + \frac{4r}{3}\right)$$

Rearranging:

$$\begin{aligned}h &= \frac{V}{\pi r^2} - \frac{4r}{3}\\ &= 9.45 \text{ mm}\end{aligned}$$

So the total length of the new tablet is $9.45 + 2(1.8) = 13.1$ mm

34 a Converting all lengths to cm for consistency of calculation:

Cylinder height $h_c = 180$

Spike height $h_s = 10$

$$\begin{aligned}\text{Cylinder: } V &= \pi r^2 h_c\\ &= \pi \times 4^2 \times 180\\ &= 2880\pi \text{ cm}^3\\ \text{Spike: } V &= \frac{1}{3}\pi r^2 h_s\\ &= \frac{1}{3}\pi \times 4^2 \times 10\\ &= \frac{160\pi}{3} \text{ cm}^3\\ \text{Hemisphere: } V &= \frac{2}{3}\pi r^3\\ &= \frac{2}{3}\pi \times 4^3\\ &= \frac{128\pi}{3} \text{ cm}^3\\ \text{Total: } V &= 2976\pi \text{ cm}^3\\ &= 9349 \text{ cm}^3\end{aligned}$$

For a thousand posts, the volume of metal required would be $9\,349\,000 \text{ cm}^3 = 9.35 \text{ m}^3$

b Paint volume needed could be calculated by multiplying the surface area of the entire shape by 0.44 mm:

Unpainted Spike: slope length $l = \sqrt{10^2 + 4^2} = 10.8$ cm

So unpainted spike SA $= \pi r l = 4\pi \times 10.8 = 135.3$ cm^2

Unpainted cylinder SA $= 2\pi r h = 2\pi \times 4 \times 180 = 4523.9$ cm^2

Unpainted hemisphere SA $= 2\pi r^2 = 2\pi \times 4^2 = 100.5$ cm^2

Total unpainted SA $= 4759.8$ cm^2

Then approximate volume of paint is $4759.8 \times 0.04 = 190$ cm^3

This method is imprecise as it takes no account of how the paint accommodates to angles surfaces, but given the shape concerned, this is immaterial.

Alternatively, we can estimate the volume of paint as the increase in volume when the original calculations use $r = 4.04$ instead of 4 and spike length 10.04 instead of 10.

Total shape volume, from part **a**, is given by

$$V_{unpainted} = \pi r^2 h_c + \frac{1}{3}\pi r^2 h_s + \frac{2}{3}\pi r^3 = 9349 \text{ cm}^3$$

Using $r = 4.04$, $h_c = 180$ and $h_s = 10.04$, this gives

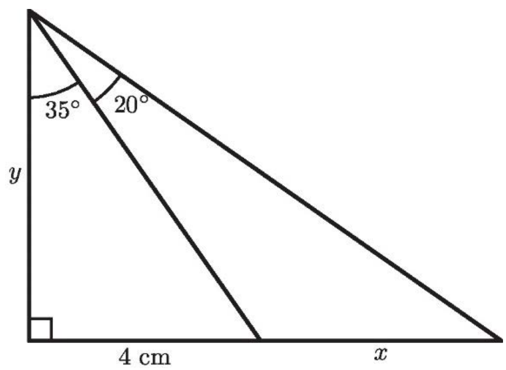
$V_{painted} = 9539.4$

The difference is accounted for by the paint, so the approximate volume of paint is 190 cm^3

For 1000 posts, the total paint needed is $190000 \text{ cm}^3 = 0.190 \text{ m}^3$

Exercise 5B

34

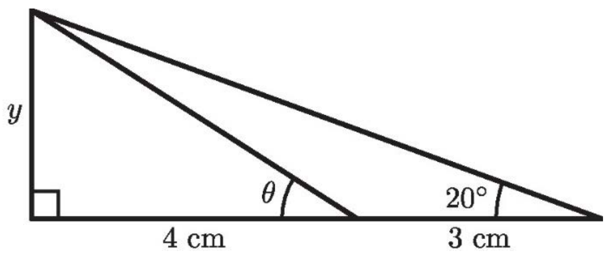


$$y = \frac{4}{\tan 35°}$$

$$x + 4 = y \tan 55°$$

$$x = \frac{4 \tan 55°}{\tan 35°} - 4$$
$$= 4.16 \text{ cm}$$

35



$y = 7 \tan 20°$

$\theta = \tan^{-1}\left(\frac{y}{4}\right)$

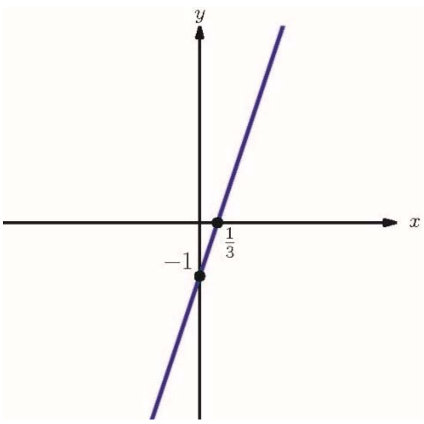
$= \tan^{-1}\left(\frac{7 \tan 20°}{4}\right)$

$= 32.5°$

36 a $(0,3)$ and $(-6,0)$

b $\tan^{-1}\left(\frac{3}{6}\right) = 26.6°$

37 a



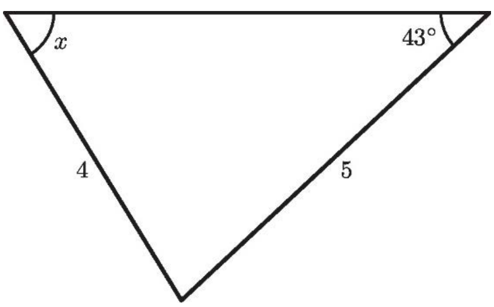
b $\tan^{-1}(3) = 71.6°$

38 Cosine Rule:

$$a^2 = 11^2 + 12^2 - 2(11)(12)\cos 35° = 48.7$$

$$a = \sqrt{48.7} = 6.98 \text{ cm}$$

39



Sine Rule:

$$\frac{\sin x}{5} = \frac{\sin 43°}{4}$$
$$x = \sin^{-1}\left(\frac{5\sin 43°}{4}\right)$$
$$= 58.5°$$

Third angle $= 180° - 58.5° - 43° = 78.5°$

40 Cosine Rule:

$$C\hat{A}B = \cos^{-1}\left(\frac{9^2 + 16^2 - 18^2}{2(9)(16)}\right)$$
$$= 87.4°$$

41 **a** Cosine Rule:

$$\hat{C} = \cos^{-1}\left(\frac{10 + 20^2 - 11^2}{2(10)(20)}\right)$$
$$= 18.6°$$

b Sine Rule for area:

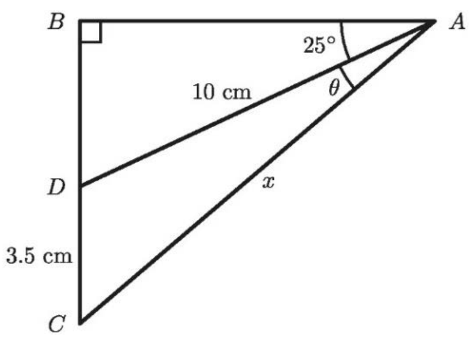
$$\text{Area} = \frac{1}{2}(10)(20)\sin 18.6°$$
$$= 32.0$$

42 Sine Rule for area:

$$\text{Area} = \frac{1}{2}(AB)(27)\sin 70° = 241$$

$$AB = \frac{482}{27\sin 70°} = 19.0$$

43



$$BD = 10\sin 25° = 4.226\ldots$$

$$AB = 10\cos 25° = 9.063\ldots$$

$$x = \sqrt{(BA)^2 + (BC)^2}$$
$$= \sqrt{9.06\ldots^2 + 7.72\ldots^2}$$
$$= 11.8\text{ cm}$$

$$\theta = \cos^{-1}\left(\frac{AB}{x}\right) - 25$$
$$= 15.5°$$

44 y-intercept is 8, crosses x-axis at 10

Angle with the x-axis is $\tan^{-1}\left(\frac{8}{10}\right) = 38.7°$

45 **a** Intersection: $2x - 8 = \frac{1}{4}x - 1$

$$\frac{7}{4}x = 7$$
$$x = 4$$

The lines intersect at $(4,0)$

b Angle between the lines:

Angle of $y = 2x - 8$ is $\tan^{-1}(2) = 63.4°$

Angle of $y = \frac{1}{4}x - 1$ is $\tan^{-1}\left(\frac{1}{4}\right) = 14.0°$

Angle between the lines is therefore $63.4 - 14.0 = 49.4°$

46 Line $2x - 5y = 7$ has gradient $\frac{2}{5}$ and angle to the horizontal $\tan^{-1}\left(\frac{2}{5}\right) = 21.8°$

Line $4x + y = 8$ has gradient -4 and angle to the horizontal $\tan^{-1}(-4) = -76.0°$

Angle between the lines is therefore $21.8 — 76.0 = 97.8°$

Acute angle is $82.2°$

47 Sine Rule:

$$\frac{b}{\sin 60°} = \frac{12}{\sin 40°}$$
$$b = \frac{12 \sin 60°}{\sin 40°} = 16.2 \text{ cm}$$

48 Cosine Rule:

$$a = \sqrt{5^2 + 8^2 - 2(5)(8)\cos 45°}$$
$$= 5.69 \text{ cm}$$

49 Cosine Rule:

$$A = \cos^{-1}\left(\frac{6^2 + 8^2 - 4^2}{2(6)(8)}\right)$$
$$= 29.0°$$

50 Sine Rule:

$$\frac{\sin Y}{8} = \frac{\sin 66°}{10}$$
$$y = \sin^{-1}\left(\frac{8 \sin 66°}{10}\right)$$
$$= 47.0°$$

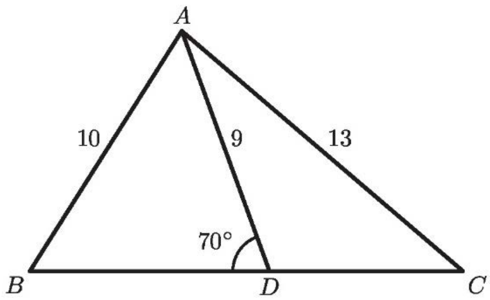
52 $A = 180 - 32 - 64 = 84°$

Sine Rule:

$$\frac{a}{\sin 84°} = \frac{3}{\sin 32°}$$

$$a = \frac{3 \sin 84°}{\sin 32°}$$
$$= 5.63 \text{ cm}$$

53



Sine Rule in ABD

$$A\hat{B}D = \sin^{-1}\left(\frac{9 \sin 70°}{10}\right) = 57.7°$$

Sine Rule in ACD

$$A\hat{C}D = \sin^{-1}\left(\frac{9 \sin 110°}{13}\right) = 40.6°$$

Then $B\hat{A}C = 180 - 57.7 - 40.6 = 81.7°$

Cosine Rule in ABC

$$BC = \sqrt{10^2 + 13^2 - 2(10)(13)\cos 81.7°}$$
$$= 15.2$$

54 Sine Rule for area:

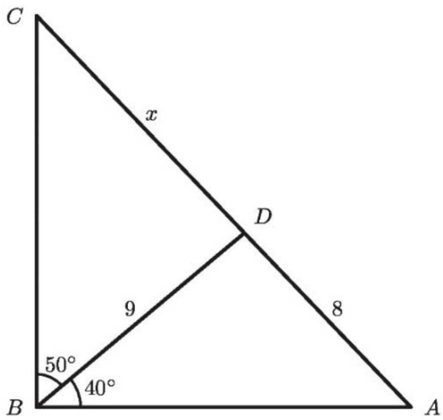
$$\text{Area} = \frac{1}{2}(6)(11)\sin\theta = 26$$

$$\theta = \sin^{-1}\left(\frac{52}{66}\right) = 52.0°$$

Cosine Rule:

$$AB = \sqrt{6^2 + 11^2 - 2(6)(11)\cos\theta}$$
$$= 8.70$$

55



Sine Rule in ABD:

$$\hat{B} = \sin^{-1}\left(\frac{9\sin 40°}{8}\right) = 46.3°$$

$$\hat{C} = 90 - 46.3 = 43.7°$$

Sine Rule in ACD:

$$x = \frac{9\sin 50°}{\sin\hat{C}} = 9.98$$

56 Let the base length be b.

Then $h = b\tan 40°$ and $d + h = b\tan 50°$

$$h = b\tan 50° - d$$
$$= \frac{h}{\tan 40°}\tan 50° - d$$

Rearranging:

$$h\left(\frac{\tan 50°}{\tan 40°} - 1\right) = d$$

$$h = \frac{d\tan 40°}{\tan 50° - \tan 40°}$$

57 a $x = \frac{h}{\tan 30°}, y = \frac{4-h}{\tan 10°}$

b Since $x + y = 8$

$$h\left(\frac{1}{\tan 30°} - \frac{1}{\tan 10°}\right) + \frac{4}{\tan 10°} = 8$$

$$h = \frac{8 - \frac{4}{\tan 10°}}{\left(\frac{1}{\tan 30°} - \frac{1}{\tan 10°}\right)} = 3.73$$

Exercise 5C

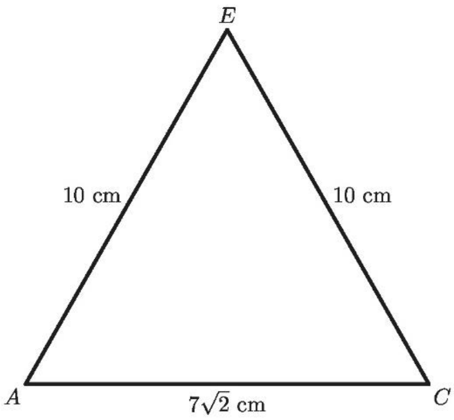
20 $h = 40\tan 55° = 57.1$ m

21 a $HB = \sqrt{5^2 + 12^2 + 9^2} = 15.8$

b $BD = \sqrt{12^2 + 9^2} = 15$

$$H\hat{B}D = \cos^{-1}\left(\frac{BD}{HB}\right) = 18.4°$$

22 a



b Let X be the centre of the base, midpoint of AC.

$$AX = \frac{7\sqrt{2}}{2}$$

The pyramid height $EX = \sqrt{EA^2 - AX^2} = \sqrt{100 - 24.5} = 8.69$ cm

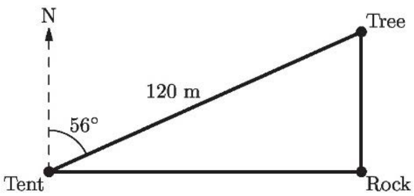
c Cosine Rule:

$$A\hat{E}C = \cos^{-1}\left(\frac{10^2 + 10^2 - (7\sqrt{2})^2}{2(10)(10)}\right) = 59.3°$$

23 Radius = 6

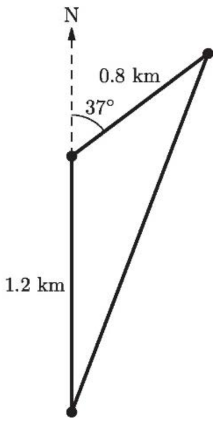
$$\theta = \tan^{-1}\left(\frac{9}{6}\right) = 56.3°$$

24 a



b $RT = 120\cos 56° = 67.1$ m

25 **a**



b Distance east of the port is $0.8 \sin 37° = 0.481$ km

Distance north of the port is $1.2 + 0.8 \cos 37° = 1.84$ km

Total distance from port is $\sqrt{1.2^2 + 0.481^2} = 1.90$ km

26 $BT = 1.6 + 6.5 \tan 62° = 13.8$ m

27 $d = 9 \tan 78° = 42.3$ m

28 If the initial distance from the statue is x and the height of the statue is h

$x \tan 17.7° = h \quad (1)$

$(x + 5) \tan 12.0° = h \quad (2)$

$$(1): x = \frac{h}{\tan 17.7°}$$

$$(2): h\left(1 - \frac{\tan 12.0°}{\tan 17.7°}\right) = 5 \tan 12.0°$$

$$h = \frac{5 \tan 12.0°}{\left(1 - \frac{\tan 12.0°}{\tan 17.7°}\right)} = 3.18 \text{ m}$$

29 If the height of the lighthouse is h and the distance of the first buoy from the lighthouse is x then

$h = x \tan 42.5° \quad (1)$

$h = \sqrt{x^2 + 18^2} \tan 41.3° \quad (2)$

$$(1): x = \frac{h}{\tan 42.5°}$$

$$(2): \left(\frac{h}{\tan 41.3°}\right)^2 = \left(\frac{h}{\tan 42.5°}\right)^2 + 324$$

$$h^2 = \frac{324}{\frac{1}{\tan^2 41.3°} - \frac{1}{\tan^2 42.5°}}$$

$$h = \sqrt{\frac{324}{\frac{1}{\tan^2 41.3°} - \frac{1}{\tan^2 42.5°}}} = 55.6 \approx 56 \text{ m}$$

30 a $AG = \sqrt{7^2 + 4^2 + 9^2} = 12.1$ cm

b $AC = \sqrt{4^2 + 9^2} = 9.85$ cm

$C\hat{A}G = \cos^{-1}\left(\frac{9.85}{12.1}\right) = 35.4°$

c $B\hat{A}G = \cos^{-1}\left(\frac{9}{12.1}\right) = 41.9°$

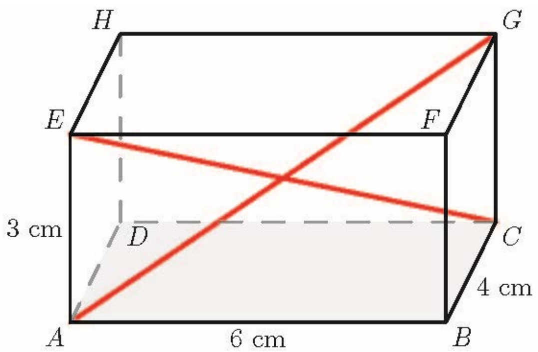
31 a

$$AG = \sqrt{AB^2 + AD^2 + AE^2}$$
$$AC^2 = AB^2 + AD^2 = 13^2$$
$$AF^2 = AB^2 + AE^2 = 7^2$$
$$CF^2 = AH^2 = AD^2 + AE^2 = 11^2$$
$$AB^2 + AD^2 + AE^2 = \frac{1}{2}(AC^2 + AF^2 + CF^2) = \frac{339}{2}$$
$$AG = \sqrt{\frac{339}{2}} = 13.02 \text{ m}$$

b

$$C\hat{A}G = \cos^{-1}\left(\frac{AC}{AG}\right)$$
$$= \cos^{-1}\left(\frac{13\sqrt{2}}{\sqrt{339}}\right)$$
$$= 3.11°$$

32

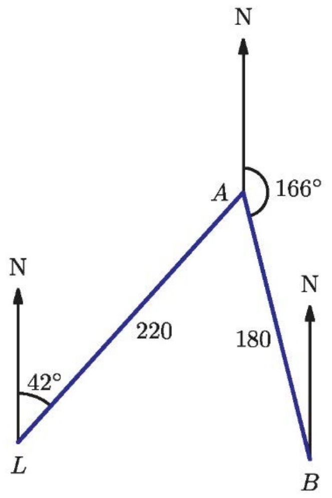


a $AG = CE = \sqrt{3^2 + 6^2 + 4^2} = 7.81$ cm

b The two lines cross in the centre of the cuboid at point X, so CGX is an isosceles, with sides 3.91 and base 3.

$$C\hat{X}G = \cos^{-1}\left(\frac{3.91^2 + 3.91^2 - 3^2}{2(3.91)(3.91)}\right)$$
$$= 45.2°$$

33



The dog runs from lighthouse L to A and then from A to B.

$$L\hat{A}B = 42° + (180 - 166)° = 56°$$

Cosine Rule in LAB:

$$LB = \sqrt{220^2 + 180^2 - 2(220)(180)\cos 56°} = 191 \text{ m}$$

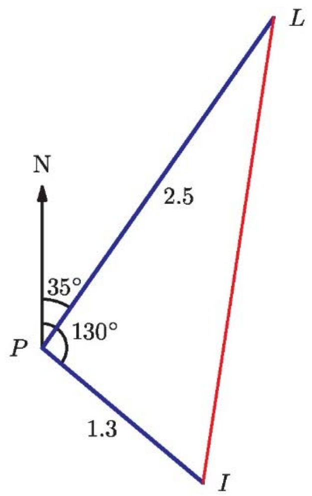
Sine Rule:

$$A\hat{L}B = \sin^{-1}\left(\frac{180 \sin 56°}{191}\right) = 51.3°$$

The bearing from L to B is $51.3 + 42 = 93.3°$

So the bearing for the return journey from B to L is $273°$

34 If the lighthouse is at L, the port at P and the island at I then



$$L\hat{P}I = 130 - 35 = 95°$$

Cosine Rule:

$$LI = \sqrt{2.5^2 + 1.3^2 - 2(2.5)(1.3)\cos 95°} = 2.92 \text{ km}$$

Sine Rule:

$$L\hat{I}P = \sin^{-1}\left(\frac{2.5\sin 95°}{2.92}\right) = 58.64°$$

The bearing from I to L is $(58.64 + 130 - 180) = 8.64°$

35 If the apex is at A, centre of base at X and the centre of one side Q

$$QX = 115 \text{ m}$$

$$A\hat{Q}X = 42°$$

So height of pyramid $AX = 115\tan 42° = 104$ m

36 a $BT = 19.5\tan 26° = 9.51$ m

b $d = \frac{BT}{\tan 41°} = 10.9$ m

c Cosine Rule:

$$R\hat{B}M = \cos^{-1}\left(\frac{19.5^2 + 10.9^2 - 14.7^2}{2(19.5)(10.9)}\right) = 48.3 \approx 48°$$

37 Let the distance of the base of the painting from the floor be x and the height of the painting be h.

$$x = 2.4\tan 55°$$

$$x + h = 2.4\tan 72°$$

So $h = 2.4(\tan 72° - \tan 55°) = 3.96$ m

38 a Volume of a square based pyramid is given by $V = \frac{1}{3}b^2h$ where $b$is the length of the sides of the base and h is the pyramid's vertical height.

Therefore, the volume of the Louvre pyramid is $V = \frac{1}{3}(34^2)(21.6) = 8323.2 \ldots m^3$.

We require one unit per $1000m^3$

Hence, 9 units are required.

b Note that we only have to consider the heat loss through the four glass sides exposed to the outside air.

The surface area of the pyramid through which heat is lost is given by

$$SA = 2a\sqrt{\frac{a^2}{4} + h^2} = 2(34)\sqrt{\frac{34^2}{4} + 21.6^2} = 1869.14 \ldots \text{m}^2$$

And the power required to offset this heat loss is simply
$192 \times 1869.14 \ldots = 359000 \text{ W} = 359 \text{ KW}$

c Consider the triangle formed by the base of the pyramid, the vertical to the pyramid's peak, and the side of the pyramid.

We can apply trig. to this triangle to find the elevation, θ.

$$\tan\theta = \frac{21.6}{0.5 \times 34}$$

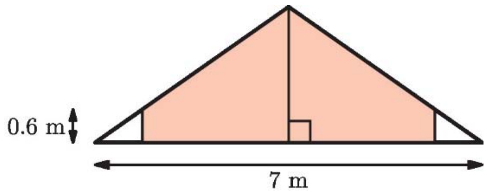
Hence, the elevation is 51.9 and therefore scaffolding is required.

39 a We have a maximum height h when $h = \frac{7}{2}\tan 35° = 2.45$ m

b The area of the face of the prism is given by $A = \frac{1}{2}(7)(2.45) = 8.58\text{ m}^2$

And so, the volume of the roof space is $V = \frac{1}{2}(7)(2.45)(5) = 42.9\text{ m}^3$

c



Using proportion: If the maximum height is 2.45 m then the proportion of the width which is over 0.6 m is $1 - \left(\frac{0.6}{2.45}\right) = 75.5\%$

d The non-usable volume will be contained in the two identical triangular prisms at the edge of the roof. Each has height 0.6 m and width $(1 - 0.755) \times 3.5 = 0.857$ m so has cross-sectional area $0.6 \times 0.857 = 0.514\text{ m}^2$

The proportion of the volume that is unusable is therefore $\frac{0.514}{8.58} = 5.99\%$

The proportion of the volume that is usable is 94.0%

(Alternatively, it can be seen that the non-usable triangles together form an isosceles similar to the overall triangle; since scale factor from whole triangle to unused part is 24.5%, it follows that the proportion of the cross-sectional area not used, and hence the proportion of the prism volume not used, will be $(24.5\%)^2 = 5.99\%$. Again, the proportion of the volume that is usable is 94.0%.)

40 $BE = \sqrt{7^2 + 9^2} = \sqrt{130}$

$BG = \sqrt{4^2 + 7^2} = \sqrt{65}$

$EG = \sqrt{4^2 + 9^2} = \sqrt{97}$

Cosine Rule:

$$B\hat{E}G = \cos^{-1}\left(\frac{130 + 97 - 65}{2\sqrt{130}\sqrt{97}}\right) = 43.8°$$

Sine Rule for area:

$$\text{Area } BEG = \frac{1}{2}\sqrt{130}\sqrt{97}\sin 43.8° = 38.9\text{ cm}^2$$

41 ABV is isosceles with equal sides 23 cm and base 20 cm. Its altitude $MV = \sqrt{23^2 - 10^2} = 20.7$ cm

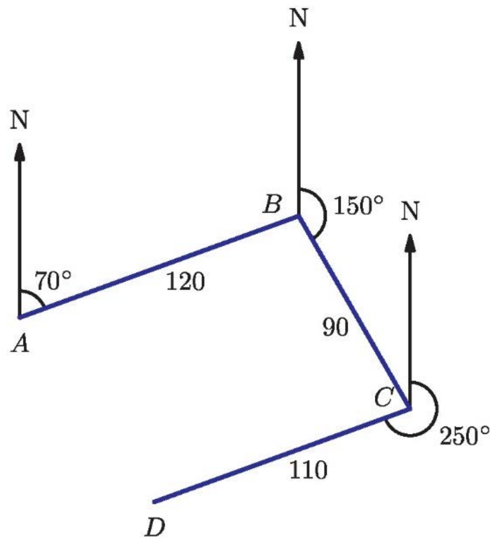
Since the pyramid has a square base, $NV = MV$ and MNV is isosceles.

$$MN = \sqrt{200}\text{ cm}$$

Cosine Rule:

$$M\hat{V}N = \cos^{-1}\left(\frac{20.7^2 + 20.7^2 - 200}{2(20.7)(20.7)}\right) = 39.9°$$

42



If Amy starts at A, then travels to B, C and finally D:

$$A\hat{B}C = 70 + 180 - 150 = 100°$$

Cosine Rule in ABC:

$$\begin{aligned} AC &= \sqrt{AB^2 + BC^2 - 2(AB)(BC)\cos A\hat{B}C} \\ &= \sqrt{120^2 + 90^2 - 2(120)(90)\cos 100°} \\ &= 162 \text{ m} \end{aligned}$$

Sine Rule in ABC:

$$A\hat{C}B = \sin^{-1}\left(\frac{120 \sin 100°}{162}\right) = 46.8°$$

Then $A\hat{C}D = 360 - 250 - 30 - 46.8 = 33.2°$

Cosine Rule in ACD:

$$AD = \sqrt{110^2 + 162^2 - 2(110)(162)\cos 33.2°} = 92.3 \text{ m}$$

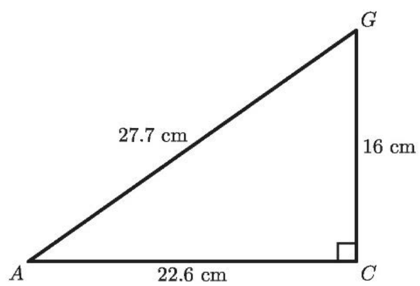
Mixed Practice

1 $h = 50 \tan 35° = 35.0$ m

2 **a** $AC = 16\sqrt{2} = 22.6$ cm

$AG = 16\sqrt{3} = 27.7$ cm

b



c $C\hat{A}G = \tan^{-1}\left(\frac{16}{16\sqrt{2}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 35.3°$

3 a $AC = 23\sqrt{2} = 32.5$ cm

b If X is the centre of the base, then $AX = 0.5AC = 16.3$ cm

$XE = 16.3 \tan 56° = 24.1$ cm

c $AE = \frac{16.3}{\cos 56°} = 29.1$ cm

4 $\theta = 2\tan^{-1}\left(\frac{5}{12}\right) = 45.2°$

5 a $(0,6)$

b Gradient $= \frac{2-5}{8-2} = -\frac{1}{2} = -0.5$

c $\theta = \tan^{-1}(0.5) = 26.6°$

6 a Sine Rule in ABC:

$$AC = \frac{10 \sin 100°}{\sin 50°} = 12.9 \text{ cm}$$

b Cosine Rule in ADC:

$$A\hat{D}C = \cos^{-1}\left(\frac{7^2 + 12^2 - 12.9^2}{2(7)(12)}\right) = 80.5°$$

7 Sine Rule:

$$b = \frac{10 \sin 70°}{\sin 50°} = 12.3 \text{ cm}$$

8 Cosine Rule:

$$a = \sqrt{8^2 + 10^2 - 2(8)(10)\cos 15°} = 3.07 \text{ cm}$$

9 Cosine Rule:

$$A = \cos^{-1}\left(\frac{5^2 + 7^2 - 3^2}{2(5)(7)}\right) = 21.8°$$

10 Sine Rule:

$$Y = \sin^{-1}\left(\frac{12 \sin 42°}{15}\right) = 32.4°$$

11 Sine Rule:

$$Q = \sin^{-1}\left(\frac{4 \sin 120°}{9}\right) = 22.6°$$

So $R = 180 - 120 - 22.6 = 37.4°$

12 $A = 180 - 32 - 72 = 76°$

Sine Rule:

$$a = \frac{10 \sin 76°}{\sin 32°} = 18.3 \text{ cm}$$

13 If the initial distance from the base of the tower is x then

$x \tan 47.7° = h$ (1)

$(x + 20) \tan 38.2° = h$ (2)

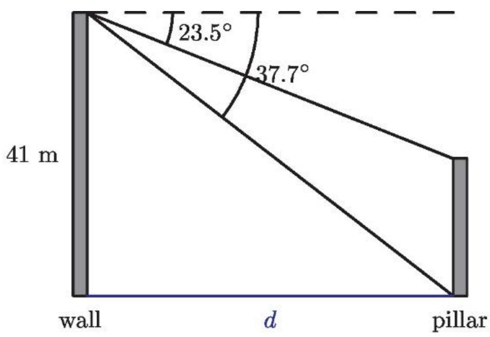
$$(1): x = \frac{h}{\tan 47.7°}$$

Substituting into (2):

$$h\left(1 - \frac{\tan 38.2°}{\tan 47.7°}\right) = 20 \tan 38.2°$$

$$h = \frac{20 \tan 38.2°}{\left(1 - \frac{\tan 38.2°}{\tan 47.7°}\right)} = 55.4 \text{ m}$$

14



The distance between wall and pillar is d

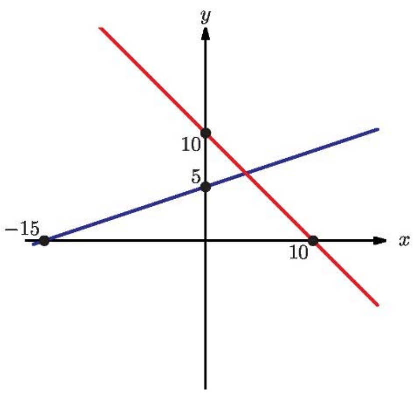
$$d = \frac{41}{\tan(37.7°)} = 53.0 \text{ m}$$

The height of the pillar is h so the difference in height between wall and pillar is $d - h$

$$d - h = 53.0 \tan 23.5° = 23.1 \text{ m}$$

So the height of the pillar is $41 - 23.1 = 17.9 \approx 18$ m

15 a



b $y_1 = 10 - x, y_2 = \frac{1}{3}x + 5$

Intersection where $10 - x = \frac{1}{3}x + 5$

$$\frac{4}{3}x = 5$$
$$x = \frac{15}{4} = 3.75$$

Point is $(3.75, 6.25)$

c Angle between y_1 and the x-axis is $\tan^{-1}\left(\frac{1}{3}\right) = 18.4°$

Angle between y_2 and the x-axis is $\tan^{-1}(-1) = -45°$

Angle between the two lines is $18.4 — 45 = 63.4°$

16 If the apex of the pyramid is E, and the base is $ABCD$ with centre X

$$AE = 26 \text{ cm}$$
$$A\hat{E}X = 35° \text{ and } A\hat{X}E = 90°$$

$$AX = 26 \tan 35° = 18.2 \text{ cm}$$
$$AB = 18.2\sqrt{2} = 25.7 \text{ cm}$$

$$\text{Vol} = \frac{1}{3}\text{base} \times \text{height}$$
$$= \frac{1}{3}(25.7^2)(26) = 5740 \text{ cm}^3$$

17 a $AM = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$

$$A\hat{M}E = \tan^{-1}\left(\frac{15}{\sqrt{45}}\right) = \tan^{-1}(\sqrt{5}) = 65.9°$$

b HME is isosceles with sides $EM = \sqrt{15^2 + 45} = \sqrt{270}$ and base 6

Cosine Rule:

$$H\hat{M}E = \cos^{-1}\left(\frac{270 + 270 - 6^2}{2\sqrt{270}\sqrt{270}}\right) = \cos^{-1}\left(\frac{504}{540}\right) = 21.0°$$

18 Cosine Rule:

$$(x + 4)^2 = x^2 + (2x)^2 - 2(x)(2x)\cos 60°$$
$$x^2 + 8x + 16 = x^2 + 4x^2 - 2x^2$$
$$2x^2 - 8x - 16 = 0$$
$$x^2 - 4x - 8 = 0$$
$$x = 2 \pm \sqrt{12}$$

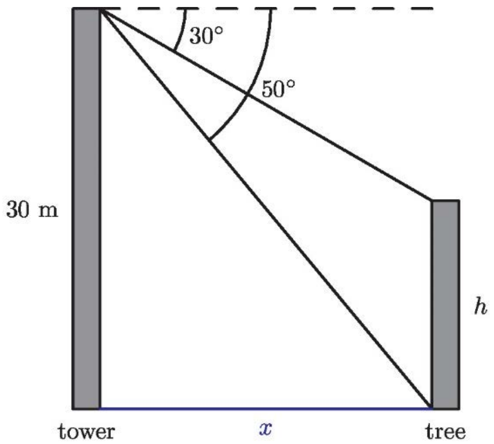
In context, $x > 0$ so the only solution is $x = 2 + \sqrt{12} = 2 + 2\sqrt{3} = 5.46$

19 Sine Rule for area:

$$84 = \frac{1}{2}x(x-5)\sin 150°$$
$$84 = \frac{1}{4}x(x-5)$$
$$x^2 - 5x - 336 = 0$$
$$(x-21)(x+16) = 0$$

In context, $x > 5$ so the only solution is $x = 21$

20



Let x be the distance between the tree and the tower on the ground.

$x = 30\tan 40° = 25.2$ m

Let h be the height of the tree.

$h = x\tan 35° = 17.6$ m

21 a $V_{ball} = \frac{4}{3}\pi(3.15)^3 = 131\text{ cm}^3$

b $V_{tube} = \pi(3.2)^2 \times 26 = 836\text{ cm}^3$

$V_{cube} - 4V_{ball} = 313\text{ cm}^3$

22 a $AB = \sqrt{9.5^2 - 8^2} = \sqrt{26.25} = 5.12$ cm

b $EF = AD = AB$

$FM = \frac{1}{2}EF = 2.56$ cm

$BM = \sqrt{BF^2 + FM^2} = \sqrt{96.8125} = 9.84$ cm

c $AM = \sqrt{AF^2 + FM^2} = \sqrt{70.5625} = 8.40$ cm

$A\hat{M}B = \cos^{-1}\left(\frac{AM}{BM}\right) = \cos^{-1}\left(\frac{8.40}{9.84}\right) = 31.4°$

23 Part A

a $XM = 5$ cm

b $VM = \sqrt{XM^2 + VX^2} = \sqrt{5^2 + 8^2} = \sqrt{89} = 9.43$ cm

c $V\hat{M}X = \cos^{-1}\left(\frac{XM}{VM}\right) = \cos^{-1}\left(\frac{5}{\sqrt{89}}\right) = 58.0°$

Part B

a Cosine Rule:

$$d = \sqrt{290^2 + 550^2 - 2(290)(550)\cos 115°} = 722 \text{ m} \approx 720 \text{ m}$$

b Sine Rule for area:

$$\text{Area} = \frac{1}{2}(290)(550)\sin 115° = 72\,300 \text{ m}^2$$

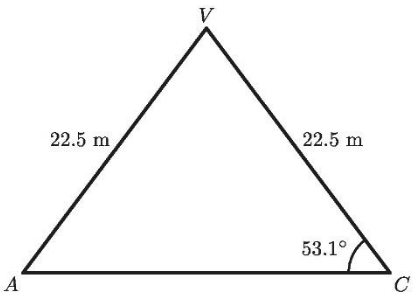
c Sine Rule

$$A\hat{B}C = \sin^{-1}\left(\frac{180 \sin 53°}{230}\right) = 38.7°$$

Then $A\hat{C}B = 180 - 53 - 38.7 = 88.3°$

24 a i 22.5 m

ii



b If the centre of $ABCD$ is X then the height of the pyramid is VX.

$V\hat{X}C = 90°$

$VX = 22.5 \sin 53.1° = 18.0$ m

c $CX = 22.5 \cos 53.1° = 13.5$ m

$AC = 2CX = 27$ m

d $BC = \frac{AC}{\sqrt{2}} = 19.1$ m

e $AP = 126 - 18.0 = 108$ m

Volume of cuboid $= AP \times AB^2 = 108 \times 19.1^2 = 39\,421 \text{ m}^3$

Volume of pyramid $= \frac{1}{3}AB^2 \times VX = \frac{1}{3} \times 19.1^2 \times 18.0 = 2187 \text{ m}^3$

Total volume $= 39\,421 + 2\,187 \approx 41\,600 \text{ m}^3$

f 90% volume $= 41\,600 \times 0.9 = 37\,500 \text{ m}^3$

Mass of $37\,500 \text{ m}^3$ air $= 37\,500 \times 1.2 = 44\,900$ kg

Worked solutions

25

Tip: If you know how to find areas using vectors this is a much easier problem to approach.

If we just use the techniques of this chapter and some basic linear algebra, we can find the intersection points, two side lengths and then find the angle between two lines, then apply the sine rule for area.

Intersection points of $y_1 = 8 - x, y_2 = 2x - 10$ and $y_3 = 12.5 - 5.5x$

y_3 & y_2:

$$\begin{aligned} 12.5 - 5.5x &= 2x - 10 \\ 7.5x &= 22.5 \end{aligned}$$

$x = 3$ so intersection is at $A(3, -4)$

y_1 & y_3:

$$\begin{aligned} 8 - x &= 12.5 - 5.5x \\ 4.5x &= 4.5 \end{aligned}$$

$x = 1$ so intersection is at $B(1,7)$

y_1 & y_2:

$$\begin{aligned} 8 - x &= 2x - 10 \\ 3x &= 18 \end{aligned}$$

$x = 6$ so intersection is at $C(6,2)$

$$AB = \sqrt{2^2 + 11^2} = 5\sqrt{5}$$

$$AC = \sqrt{3^2 + 6^2} = 3\sqrt{5}$$

Angle made by y_2 and the x-axis is $\tan^{-1}(2) = 63.4°$

Angle made by y_3 and the x-axis is $\tan^{-1}(-5.5) = -79.7°$

Angle between y_2 and y_3 is $63.4 - (-79.7) = 143° = B\hat{A}C$

$$\begin{aligned} \text{Area } ABC &= \frac{1}{2}(AB)(AC)\sin B\hat{A}C \\ &= \frac{1}{2} \times 5\sqrt{5} \times 3\sqrt{5} \sin 143° \\ &= \frac{75}{2} \times 0.6 \\ &= 22.5 \end{aligned}$$

6 Core: Statistics

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 6A

5 **a** All households in Germany

b Convenience sampling

c Households in the city may have fewer pets than in the countryside

There may be a link between pet ownership and having a child at Anke's school, such as affluence or catchment area of the school (different regions of a city have different sizes of garden or access to parks, for example).

There may even be a link between pet ownership rates and having children in the household (and since she is enquiring only from households with children, this would introduce a bias)

6 **a** The number or proportion of pupils in each year group.

b Quota sampling

c **i** Keep – this is potentially a valid result, even if it is an outlier.

ii Discard – this cannot be a true result (misrecorded, question misunderstood or answer deliberately mischievous)

7 **a** Convenience sampling

b All residents of his village

c His sample is tied to bus-stop users, and may not be representative of the village as a whole, since preferred mode of transportation is relevant to environmental attitudes.

d He would need access to all residents, and a means of requiring responses from the randomly selected individuals.

8 The observation is not necessarily correct; this feature of the sample could have occurred by chance, even with correct sampling technique.

9 **a** Continuous

b Testing the population would require running all the lightbulbs under test conditions until they failed, which would mean none would be sold.

c Listing the serial numbers in order, select every twentieth bulb.

10 **a** Quota sampling

b A stratified sample would be more representative of the scarves sold; this might be significant if colour selection were linked to gender of customer.

c Applying the percentages to the sample size 40:

30% of 40: 12 red scarves

30% of 40: 12 green scarves

25% of 40: 10 blue scarves

15% of 40: 6 white scarves

11 a Quota sampling

b The sample will be more representative of the population she believes to be in the park.

c It would be difficult to compile a list of all the animals in the park and then specifically catch and test those animals predetermined for the sample.

12 a Continuous

b The basketball team are unlikely to be representative of the students as a whole, since height is a relevant characteristic associated with membership of a basketball team

c Systematic sampling

d To be a random sample, every sample of the same size must have equal probability of being taken. However, with the sampling described, it would not be possible to select two people who are adjacent on the list; in fact, for a 'one every ten names' system, there would only be 10 possible lists, determined entirely by which student in the list Shakir began his sample at.

13 The proportions in the table are

Cat 27%, Dog 43% and Fish 30%.

Applying these proportions to the sample size, we would take 5 cats, 9 dogs and 6 fish

14 a The total number in the table is 200, so applying the same proportions to a total of 20 gives

Gender/age	12	13	14
Boys	4	5	5
Girls	0	4	2

b His sample might be representative of his school but there is no basis to believe it to be representative of the country.

(For example, if he is correct about gender and age being relevant, the fact that his sample proportions will not reflect that of the country, where 12 year old girls exist, will be a problem!)

15 a Discrete

b Not reasonably

c Convenience sampling

d Different species or genetic family lines might be clustered in different parts of the field, and if she were generalising to the country as a whole, the field itself might not be representative.

16 a i Possible, if her sample is representative.

ii Not necessarily; the school may not reflect the distribution seen nationally

iii Possible; the school might follow the national pattern and she happened to select taller students by chance.

b **i** and **ii** would have the same answer.

iii would look less likely, since her sample is now larger and less likely to be extreme by random chance.

c -32 is clearly an error, since negative height is impossible. Check the original data and if the value is clearly recorded as -32, discard it.

155 cm is very tall for a primary school student, but is theoretically possible, so should be checked and then if truly recorded as 155, the value should be retained.

17a You would expect 20 to have taken the 'say yes' and 20 to have taken 'say no', so of the subjects who had a free choice, 4 of 20 answered 'yes'. This implies a 20% 'yes' answer from the sample who had a free choice.

b **i** Out of 120 students, you would expect 20 to be answering statement 1 and 100 to be answering statement 2.

Assuming they respond honestly to the statements as instructed, we would expect 4 cheaters to be answering statement 1 and 20 cheaters to be answering statement 2.

Then $4 + 80 = 84$ would answer 'True'

ii Let the proportion who have cheated be p.

Then the number saying 'True' would equal $20p + 100(1 - p) = 48$

Rearranging:

$$\begin{aligned} 100 - 80p &= 48 \\ 80p &= 52 \\ p &= 65\% \end{aligned}$$

18 a Of the recaptured fish, $\frac{20}{40\,000}$ were labelled.

If this is a representative sample, then the proportion of labelled fish in the Sea would be

$$\frac{20}{40\,000} = \frac{50\,000}{100\,000\,000}$$

So we would estimate 100 million cod in the North Sea

b We would have to assume that all 50 000 labelled fish were still available when the sample was taken – that is that none had died (natural death or removed by predation or fishing)

We would also have to assume that there was thorough mixing of the cod population so that the sample was representative of the cod stocks as a whole. Given fish behaviour typically involves grouping in schools, this is questionable, unless the sampling was very extensive and avoided sampling largely from only a few groups.

Exercise 6B

19 a Sum of frequencies is 48

b $8 + 11 = 19$

c From GDC: $\bar{x} = 6.17$ min, $\sigma = 1.31$ min

20 a $12 + 19 = 31$

b $1.4 \leqslant m < 1.6$

c 1.34 kg

21 a Sum of frequencies $= 28$

b $15.5 \leqslant t < 17.5$

c From GDC: $\bar{t} = 17.3$ sec

This is an estimate because the actual times are not known, so the data are assumed to lie at the midpoints of their groups.

22 From GDC:

a Median $= 4.25$ min

b IQR$= 1.5$

c The second artist has songs which are longer on average and more consistent (less spread).

23 From GDC:

a Median $= 4$

b IQR $= 3$

c $Q_1 = 3$ and IQR $= 3$ so any value below $3 - 1.5(3) = -1.5$ mins would be an outlier, but there are no such values.

$Q_3 = 6$ and IQR $= 3$ so any value above $6 + 1.5(3) = 10.5$ is an outlier. 11 is an outlier.

24 a Sum of frequencies $= 67$

b 23 cm

c From GDC: mean length $= 24.7$ cm

d

$20.5 \leqslant l < 23.5$	$23.5 \leqslant l < 26.5$	$26.5 \leqslant l < 29.5$
23	30	14

e $30 + 14 = 44$

f No. The modal group is $23.5 \leqslant l < 26.5$ but the modal value is 23.

25 a Total value is $10 + 6a = 17 \times 5$

$$\begin{aligned} 10 + 6a &= 85 \\ 6a &= 75 \\ a &= 12.5 \end{aligned}$$

b From GDC, $\sigma = 13.4$

26 a Total value is $12x - 3 = 10.5 \times 6$

$$\begin{aligned} 12x - 3 &= 63 \\ 12x &= 66 \\ x &= 5.5 \end{aligned}$$

b var $= 30.9$

27 Total mark for first group: $67.5 \times 12 = 810$

Total mark for the second group: $59.3 \times 10 = 593$

Total marks over both groups: $810 + 593 = 1403$

Mean mark over both groups: $\frac{1403}{22} = 63.8$

28 Total marks required for average 60 marks: $5 \times 60 = 300$

Total marks in the first four papers: 238

She needs $300 - 238 = 62$ marks in the final paper.

29 a Mean $= \frac{2\times5+5\times8+7\times13+14a+10(a+2)}{5+8+13+14+10} = \frac{161+24a}{50}$

b $\frac{161+24a}{50} = 8.02$

$$\begin{aligned} 161 + 24a &= 401 \\ 24a &= 240 \\ a &= 10 \end{aligned}$$

30 a

$$\begin{aligned} \text{Mean shoe size} &= \frac{4.5 \times 4 + 5 \times 12 + 6.5 \times 8 + 7 \times\ 4 + 2x}{4 + 12 + 8 + 4 + 2} \\ &= \frac{158 + 2x}{30} \\ &= 5.9 \end{aligned}$$

$$\begin{aligned} 158 + 2x &= 177 \\ 2x &= 19 \\ x &= 9.5 \end{aligned}$$

b $n = 30$ so the median is midway between the 15th and 16th value, Q_1 is the 8th value and Q_3 is the 23rd value.

$Q_1 = 5, Q_2 = 5, Q_3 = 6.5$

c So IQR $= 1.5$

Values above 1.5IQR above Q_3 would be considered outliers; $6.5 + 1.5\text{IQR} = 8.75$ so $x = 9.5$ is an outlier value.

31 **a** From GDC: $\bar{x} = £440, \sigma = £268$

b Median would seem better, since there is one value substantially larger than the others which distorts the mean.

c Both will scale with the conversion:

$\bar{x} = \$(440 \times 1.31) = \$576$

$\sigma = \$(268 \times 1.31) = \$351$

32 **a** From GDC: $\bar{x} = 4, \sigma = 2.94$

b The new data values are the original values x under transformation $f(x) = 3x + 2000$

The mean will rise to $f(\bar{x})$ and the standard deviation will rise to 3σ

The new mean is $f(4) = 2012$

The new standard deviation is $3 \times 2.94 = 8.83$

33 The mean is $\frac{5}{9}51 - \frac{160}{9} = 10.6\ °\text{C}$

The standard deviation is $\frac{5}{9}3.6 = 2\ °\text{C}$

34 The mean is $5.7 \times 1.61 = 9.18$ km

The variance is $4.5 \times 1.61^2 = 11.9\ \text{km}^2$

35 New median is $-3 \times 25 = -75$

New IQR is $|-3 \times 14| = 42$

36 **a** $n = 11$ so the median is the 6th mark, Q_1 is the 3rd mark and Q_3 is the 9th mark, in an ordered list.

Since $m > 46$, that means $Q_2 = 42, Q_1 = 37$ and $Q_3 = 45$

Median is 42 and IQR is 8

b Outliers exist for values above $Q_3 + 1.5\text{IQR}$, which equals $45 + 12 = 57$.

So any value of m greater than 57 would represent an outlier.

The smallest such m is 58.

37 Suppose each test is scored out of K marks.

Total marks for the first four tests: $0.68K \times 4 = 2.72K$

Total marks required for all six tests: $0.7K \times 6 = 4.2K$

Total required in the final tests: $4.2K - 2.72K = 1.48K$

Average required in each of the final two tests: $\frac{1.48K}{2K} = 74\%$

38 **a**

$$\begin{aligned}\text{Mean grade} &= \frac{3 \times 1 + 4 \times 8 + 5 \times 15 + 6p + 7 \times 4}{1 + 9 + 14 + p + 4} \\ &= \frac{138 + 6p}{28 + p} = 5.25\end{aligned}$$

$$138 + 6p = 5.25(28 + p)$$
$$138 + 6p = 147 + 5.25p$$

$0.75p = 9$
$p = 12$

b From GDC: With $p = 12$, $\sigma = 0.968$

39 Total frequency:

$$5 + 10 + 13 + 11 + p + q = 50$$
$$p + q = 11 \qquad (1)$$

Mean:

$$\frac{3 \times 5 + 4 \times 10 + 5 \times 13 + 6 \times 11 + 7p + 8q}{50} = 5.34$$
$$186 + 7p + 8q = 267$$
$$7p + 8q = 81 \qquad (2)$$
$$(2) - 7(1): q = 4$$

Then $(1): p = 7$.

40 Median is the central value: $b = 26 \qquad (1)$

Range is 11 so $a = c - 11 \qquad (2)$

Mean is 25 so $(a + b + c) = 3 \times 25 = 75 \qquad (3)$

Substituting (1) and (2) into (3):

$2c + 15 = 75$
$c = 30$

41 Median is the central value, so $a, b \geqslant 3$ since either being less than 3 would decrease the median.

Mean is 4 so $a + b + 6 = 4 \times 5 = 20$

Then $a + b = 14$

The largest possible range occurs when one equals 3 (the lowest possible) and the other is 11.

Largest possible range is therefore 10.

42 The median of 8 values is midway between the 4th and 5th values.

Mean is 7 so $1 + 3 + 4 + 10 + 10 + 16 + x + y = 8 \times 7 = 56$

Then $x + y = 12$

The values are all integers.

We could choose $x = y = 6$, which would give a median of 6 and $x + y = 12$ as required; however, this would contradict the requirement that $x < y$.

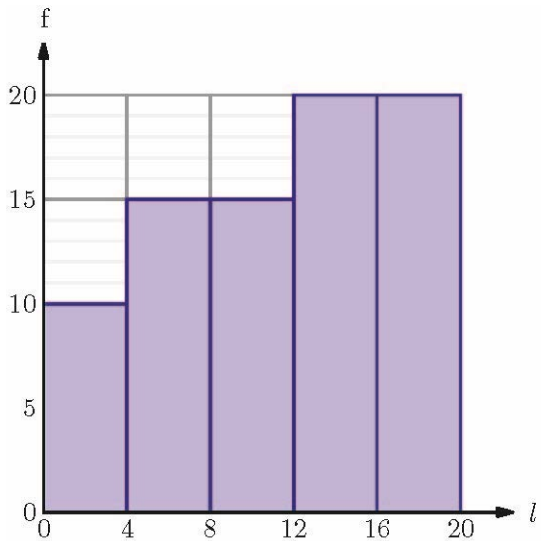
Choosing $x = 5, y = 7$ would again maintain a median of 6 and also satisfy the condition $x + y = 12$. As would the case $x = 4, y = 8$.

At $x = 3, y = 9$, x becomes the (2nd or) 3rd value, with 4 being the 4th and y the 5th, hence the median will no longer be 6.

The only possible values for (x, y) are $(5, 7)$ and $(4, 8)$.

Exercise 6C

10 a

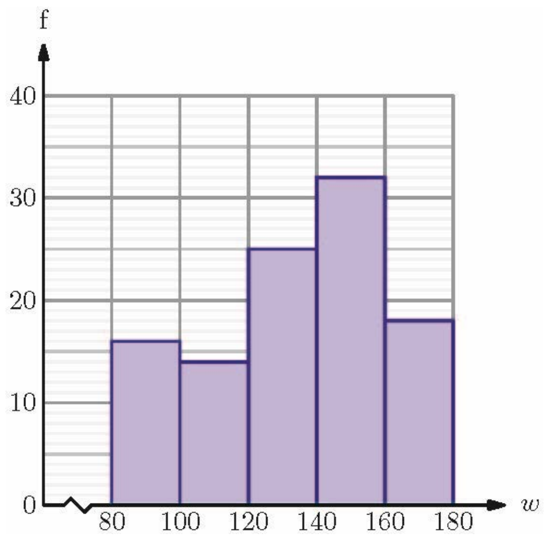


b Linear estimate: one quarter of the $12 < l \leqslant 16$ group and all the $16 < l \leqslant 20$ group:

$5 + 20 = 25$

11 a 105

b

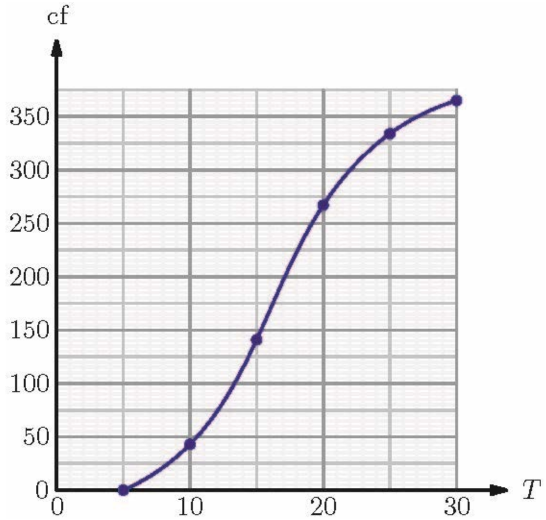


c Linear estimate: half of the $100 \leqslant m < 120$ group , all the $120 \leqslant m < 140$ group and half of the $140 \leqslant m < 160$ group:

$7 + 25 + 16 = 48$

This is $\frac{48}{105} \times 100\% = 46\%$ of the apples sampled.

12 a



b i 17 °C

ii $Q_1 \approx 13$ °C, $Q_3 \approx 20$ °C so IQR ≈ 7 °C

13 a 45

b Linear approximation: $\frac{3}{5}$ of the group $20 \leqslant t < 25$ and all the group $25 \leqslant t < 30$

$6 + 4 = 10$ pupils.

This represents $\frac{10}{45} \times 100\% = 22\%$

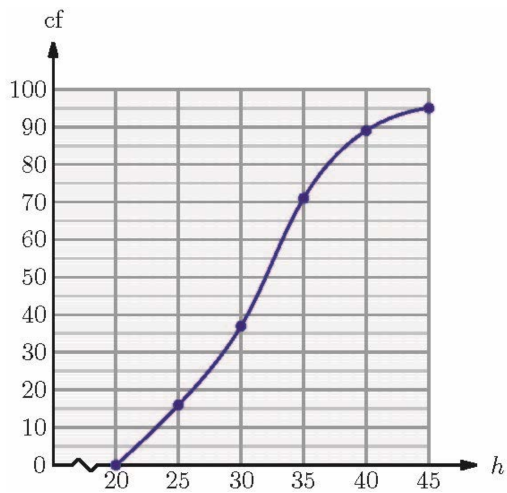
c

Time (min)	$5 \leqslant t < 10$	$10 \leqslant t < 15$	$15 \leqslant t < 20$	$20 \leqslant t < 25$	$25 \leqslant t < 30$
Freq	7	9	15	10	4

d Using midpoints of groups to estimate the mean:

$$\text{Mean} \approx \frac{7 \times 7.5 + 9 \times 12.5 + 15 \times 17.5 + 10 \times 22.5 + 4 \times 27.5}{45} = 16.9 \text{ min}$$

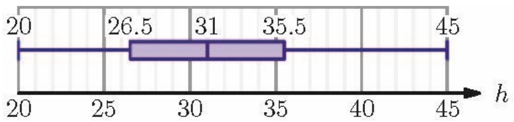
14 a



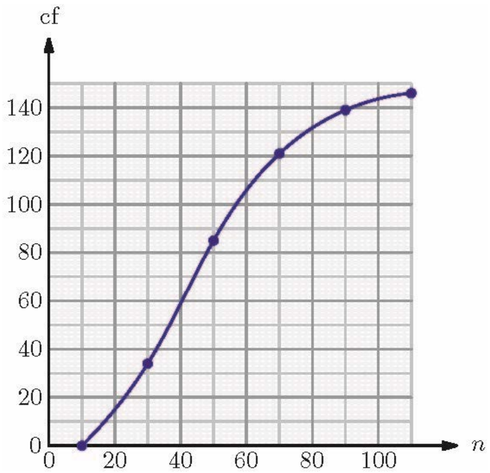
b Median ≈ 31 cm

$Q_1 \approx 26.5$ cm, $Q_3 \approx 35.5$ cm so IQR ≈ 9 cm

c



15 a

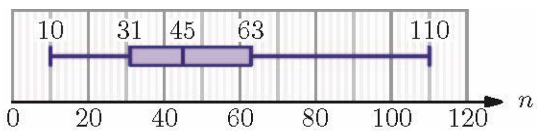


b 40

c Median ≈ 45

$Q_1 \approx 31,\ Q_3 \approx 63$ so IQR ≈ 32

d



e The distribution of number of candidates for Mathematics SL has a greater central value but is less widely spread than the distribution of number of candidates for History SL.

16 Using midpoints of groups to estimate the mean:

$$\text{Mean} \approx \frac{35 \times 145 + 60 \times 155 + 55 \times 165 + 20 \times 175}{35 + 60 + 55 + 20} = 159 \text{ cm}$$

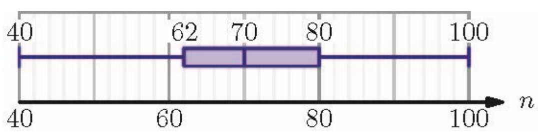
17 a 160

b Number scoring at least 55: approximately 270 so percentage passing is approximately 90%.

c 60th centile is the score exceeded by 120 students: 75 marks

d Median ≈ 72

$Q_1 \approx 62,\ Q_3 \approx 80$ so IQR ≈ 18



e The first school has a lower central score but is less widely spread (more consistent) in the scores achieved by its students, and has fewer failing the examination.

18 From GDC:

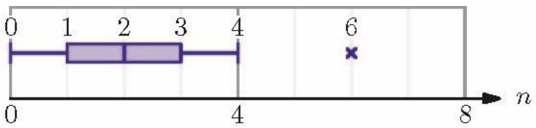
a Median, $Q_2 = 2$

$Q_1 = 1, Q_3 = 3$

b $\text{IQR} = 2$

c $Q_3 = 3$ and $\text{IQR} = 2$ so any value above $3 + 1.5(2) = 6$ would be an outlier, so 7 is an outlier.

d



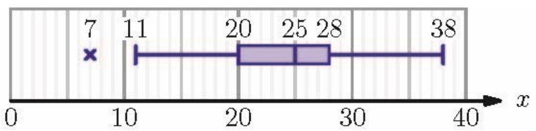
19 a $\text{IQR} = 8$

Outliers are values greater than 1.5IQR from the central 50% of the data.

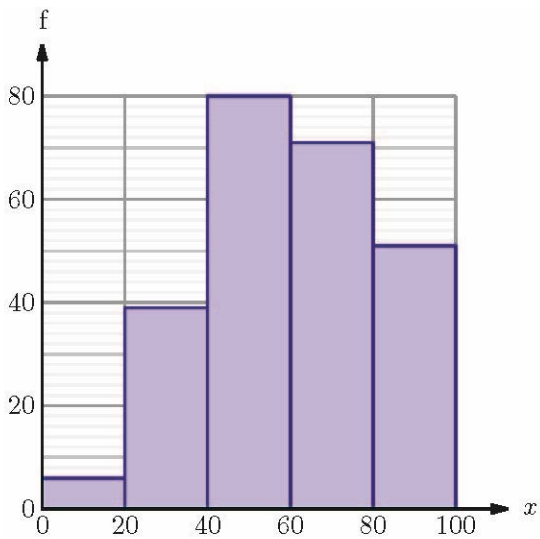
$Q_1 - 1.5 \times 8 = 8$ so 7 is an outlier.

$Q_3 + 1.5 \times 8 = 40$ so there are no outliers at the high end of the data.

b



20 a



b Using midpoints of groups to estimate the mean, and assuming the lowest possible mark is 0:

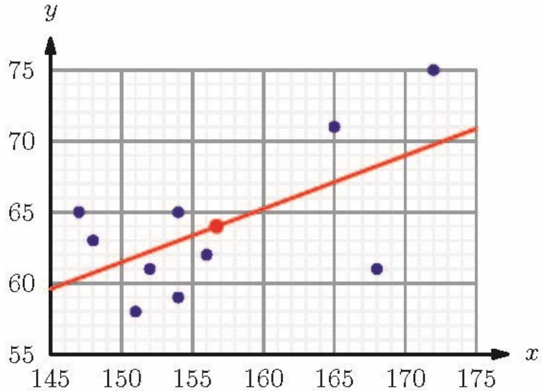
Midpoint	10	30	50	70	90
Frequency	6	39	80	71	51

$$\text{Mean} \approx \frac{6 \times 10 + 39 \times 20 + 80 \times 50 + 71 \times 70 + 51 \times 90}{247} = 59.9$$

21 A2, B3, C1

Exercise 6D

14 a, d



b The scatter growth shows a weak positive correlation

c Mean height = 156.7 cm

Mean arm length = 64 cm

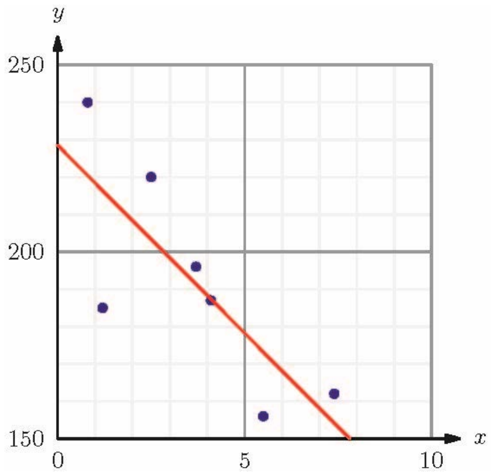
e 61.5 cm

f **i** With reservations, since the data is from 15-year olds, but it reasonable to predict arm length for a 16-year old from this data.

ii 192 cm is outside the range of the data taken, and extrapolating beyond the data would be unreliable.

iii The data is from adolescents and it would not be suitable to use the best fit line to estimate the arm length for a 72-year old.

15 a, d



b There is a fairly strong negative correlation between distance from the nearest train station and average house price; as distance increases, average house price tends to fall.

c Mean distance $= 3.6$ km

Mean price $= \$192\,000$

e Using the line to estimate: A village 6.7 km from its nearest train station would be predicted to have an average house price approximately $= \$161\,000$

16 A) Strong positive correlation: Graph 1

B) Weak negative correlation: Graph 2

C) Strong negative correlation: Graph 3

D) Weak positive correlation: Graph 4

17 a $r = 0.688$

b From GDC: $y = 0.418x + 18.1$

c The model predicts a second test score of 46.5

d The statement is not appropriate, as it infers causation from the correlation. There is no reason to believe that math score drives chemistry score (or vice versa), the correlation merely indicates that they are linked.

18 a $r = 0.745$

b Greater spending on advertising tends to yield a greater profit.

c From GDC: $y = 10.8x + 188$

d i $\$1270$

ii $\$2350$

e The estimate for \$100 is within the values of the data, and is therefore more reliable than the estimate for \$200, for which we have to extrapolate from the graph.

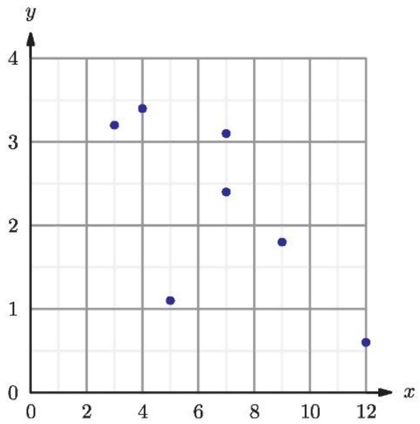
19 a $r = -0.0619$

b $y = 51.6 - 0.0370x$

c It would not be reasonable to use a mark in History to predict marks in a French test. The correlation is very low, far lower than the critical value, so the data does not indicate a significant link between the two data sets.

d As for part **c**, the correlation value is so low that no link is established between the two data sets. Neither is seen to predict the other, so a French test result could also not reliably be used to predict score in a History test.

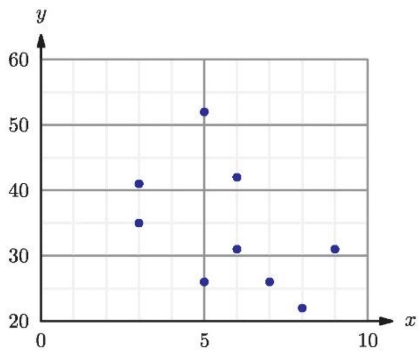
20 **a**



b $r = -0.695$

c The calculated correlation coefficient has an absolute value greater than the critical value, so we conclude that there is statistically significant negative correlation between age and value; as age increases, value tends to decrease.

21 **a**



b Weak, negative correlation

c $r = -0.480$

d The absolute value of the correlation coefficient is less than the critical value. The data does not show a significant correlation.

22 **a** There is a moderate positive correlation between head circumference and arm length in the sample.

b This head circumference is within the values of the data set and there is a moderate correlation, so the estimate can be considered reliable.

c Substituting $x = 51.7$ into the regression line equation: $y = 40.8$.

An arm length of 40.8 cm is predicted.

23 **a** $r = 0.828$

There is a moderate positive correlation between time spent practising and test mark.

b $m = 0.631t + 5.30$

Worked solutions

c The intercept value suggests that if no time is spent revising, Theo would expect to score 5.3 on a test.

The gradient suggests that for every minute spent studying, he increases his expected test score by 0.631 marks.

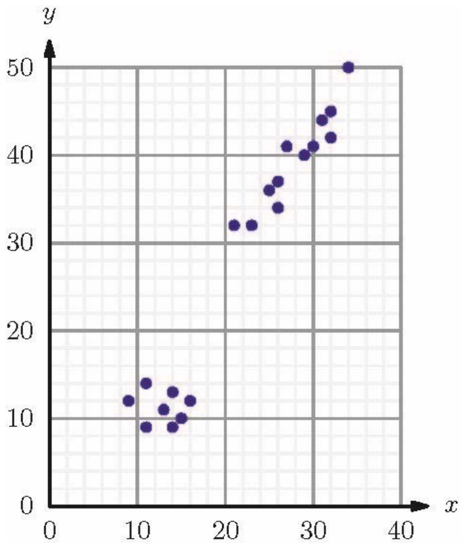
24 a The moderate positive correlation suggests that as advertising budget increases, so too does profit.

b No. The correlation shows a linkage, it does not show causation.

c i The slope 3.25 suggests that for every thousand Euros spent in advertising, the profits are expected to rise by 3 250 Euros, within the interval of the advertising budget investigated.

ii The intercept 138 suggests that with no advertising at all, there would be a profit of 138 000 Euros.

25 a



b The two regions represent sales in cold weather (perhaps winter) and sales in warm weather (summer).

c For the winter temperatures, there appears to be no significant correlation between temperature and sales.

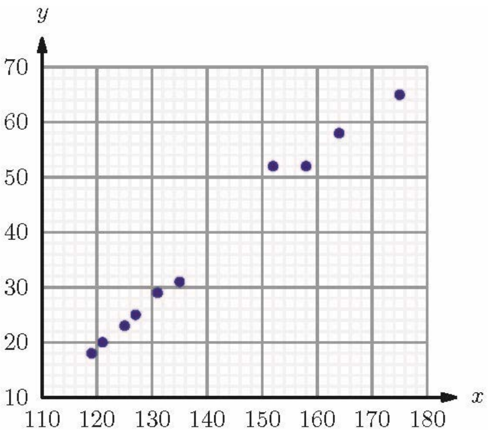
For the summer temperatures, there appears to be a strong positive correlation between temperature and sales.

d Excluding the lower population (temperatures below 20 °C), the regression line is calculated as

$$y = 2.97 + 1.30x$$

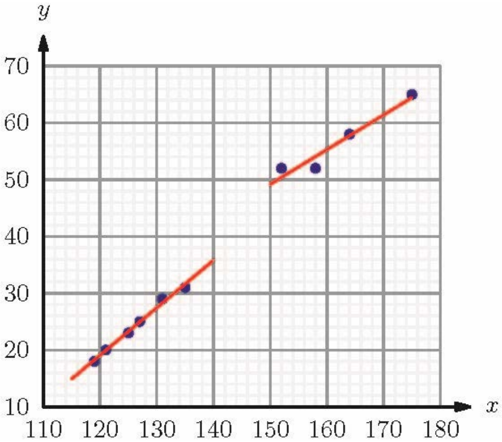
This predicts that for a temperature of 28 °C, the sales would be approximately 39.5.

26 a



b The children and the adults have height to mass relationships, but each subpopulation shows a (different) positive linear correlation.

c



children $(126, 24.3)$, adults $(162, 56.8)$

d The regression line for the childrens' data (heights less than 140 cm) in the data has equation

$$m = 0.835h - 81.1$$

This predicts a mass of 19.0 kg for a child with height 120 cm.

27 a For x:

$$Q_1 = 12, Q_3 = 24.$$

For y,

$$Q_1 = 11, Q_3 = 19$$

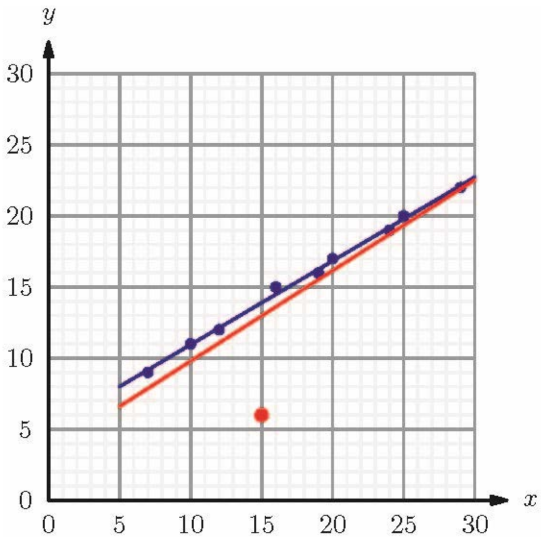
b Outliers are values more than 1.5IQR above Q_3 or below Q_1.

For x, IQR $= 12$ and all the data values lie within $(12 - 1.5(12), 24 + 1.5(12)) = (-6,42)$

For y, IQR $= 8$ and all the data values lie within $(11 - 1.5(8), 19 + 1.5(8)) = (-1,31)$

There are no outliers in either data set.

c, d, e



Mixed Practice

1 **a** **i** Systematic sampling

ii Students might have a weekly borrowing pattern, so would either be sampled every time or not at all.

Even without such a regular pattern, some days of the week might be busier than others, and picking the same day every week would then not be representative.

b **i** Each possible ten day selection has an equal chance of being picked for the sample.

ii The sample is more likely to be representative of all the days in the investigation.

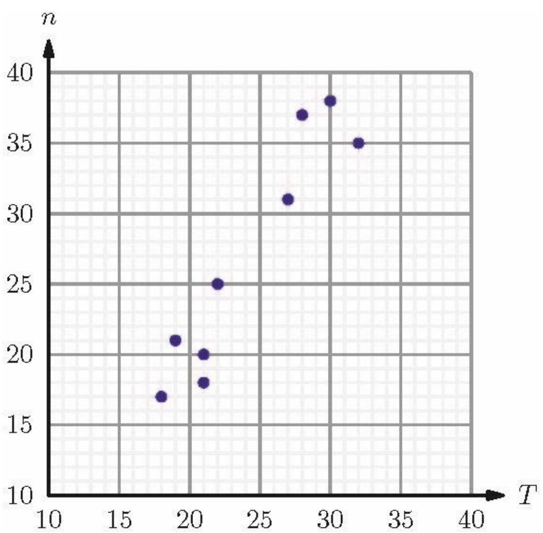
c From GDC:

i Range $= 11$

ii $\bar{x} = 17.4$

iii $\sigma = 3.17$

2 **a**



b There is a strong positive correlation; as temperature increases, sale of cold drinks also tends to increase.

c From GDC: $n = 1.56T - 10.9$

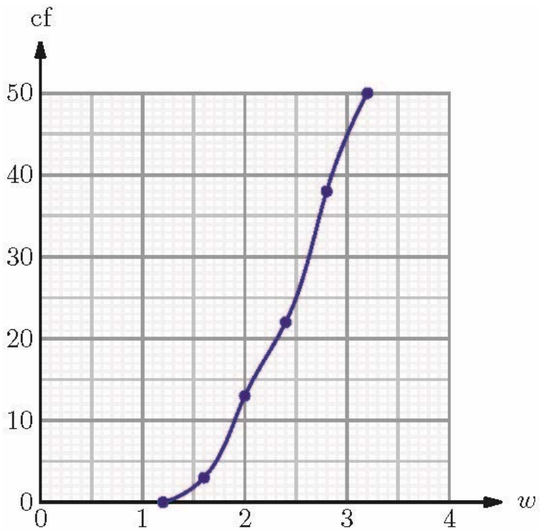
d 26 °C is within the range of data values for a high correlation line, so we can confidently estimate using the regression equation.

When $T = 26, n \approx 30.0$

3 **a** Using midpoints of groups to estimate the mean:

$$\text{Mean} \approx \frac{4 \times 1.4 + 10 \times 1.8 + 8 \times 2.2 + 16 \times 2.6 + 12 \times 3.0}{50} = 2.38 \text{ kg}$$

b

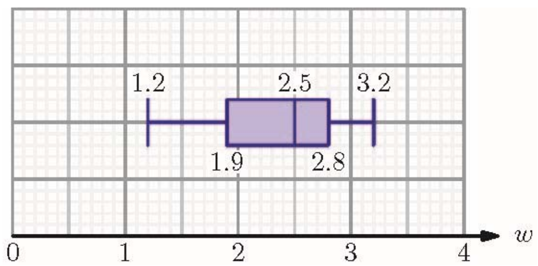


c From the graph:

Median $= 2.5$ kg

$Q_1 = 1.9$ kg, $Q_3 = 2.8$ kg so IQR $= 0.9$ kg

d



4 **a** Discrete

b Mode $= 0$

c From GDC:

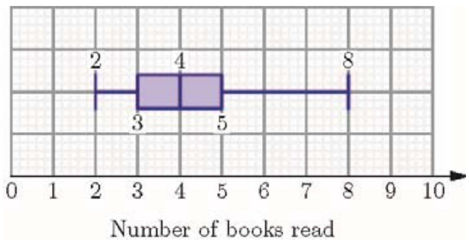
i $\bar{x} = 1.47$ passengers

ii Median $= 1.5$ passengers

iii $\sigma = 1.25$ passengers

5 **a** Median = 4

b



c $40 \times 25\% = 10$ students

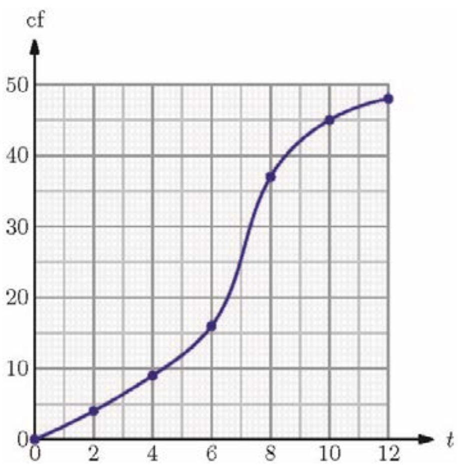
6 **a** From GD: $r = 0.996$

b From GDC: $y = 3.15x - 15.4$

c High correlation and value within the data range allows confident approximation from the regression equation.

When $x = 26, y \approx 66.5$

7 **a**



b From the graph:

i Median = 6.9 mins

ii $Q_1 = 5.0$ mins, $Q_3 = 7.9$ mins so IQR = 1.9 mins

iii 90th centile is at $cf = 43.2$, which corresponds to approximately 9.3 mins

c

Time	Freq
$0 \leqslant t < 2$	4
$2 \leqslant t < 4$	5
$4 \leqslant t < 6$	7
$6 \leqslant t < 8$	11
$8 \leqslant t < 10$	8
$10 \leqslant t < 12$	3

d Using midpoints of groups to estimate the mean:

$$\text{Mean} \approx \frac{4\times1+5\times3+7\times5+21\times7+8\times9+3\times11}{48} = 6.375 \text{ mins}$$

8 **a** From GDC:

Median $= 46$

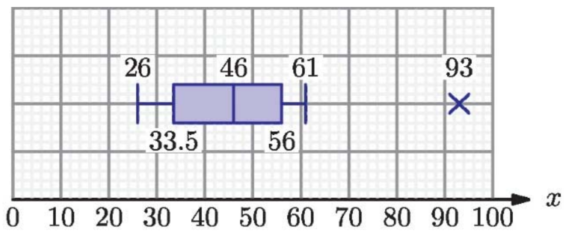
$Q_1 = 33.5, Q_3 = 56$ so IQR $= 22.5$

b Outliers are values more than 1.5IQR above Q_3 or below Q_1.

The lower boundary for outliers is $33.5 - 1.5(22.5) = -0.25$ so there are no outliers at the lower end of the data.

The upper boundary for outliers is $56 + 1.5(22.5) = 89.75$ so 93 is an outlier.

c



9 **a** From GDC:

Mean $= 121$ cm

Var $= 22.9 \text{ cm}^2$

b Adding a constant changes the mean but not the variance.

For the new data,

Mean $= 156$ cm

Var $= 22.9 \text{ cm}^2$

10 Let X be the distance travelled, in kilometres.

$\bar{x} = 11.6, \sigma_x = 12.5$

Let Y be the cost of his travels, in \$

$Y = 15 + 3.45X$

Then $\bar{y} = 15 + 3.45\bar{x} = \$55.02$

And $\sigma_y = 3.45\sigma_x = \$43.13$

11

$$\text{Mean} = \frac{5\times0+6\times1+8\times2+3x}{19+x}$$

$$= \frac{22+3x}{19+x} = 1.6$$

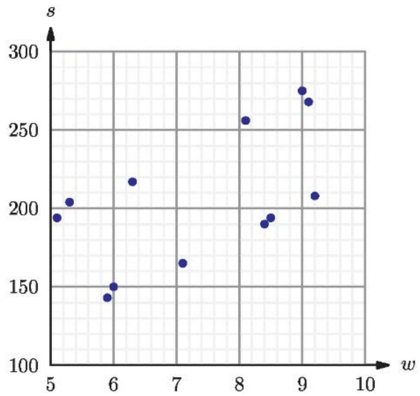
$22 + 3x = 30.4 + 1.6x$

$1.4x = 8.4$

$x = 6$

Worked solutions

12 a



b From GDC: $w = 19.1s + 99.0$

c From GDC: $r = 0.994$

This indicates a strong positive correlation between shell length and mass; as shell length increases, mass reliably increases as well.

d 8 cm is within the range of data values, and the correlation is high, so an estimate from the regression equation is reliable.

The model predicts a mass of 252 g

e The model would predict a mass of between 137 g and 175 g.

This prediction is not reliable, since these masses are outside the data gathered, and cannot reliably be estimated by extrapolating the linear model for adult crabs.

13 a i A positive y-intercept of 2 could be interpreted as a fixed increase in performance of 2 miles for all athletes, irrespective of their previous fitness levels.

ii The positive gradient of 1.2 could be interpreted as a variable 20% increase in performance levels for all athletes.

b The new mean would be $1.2(8) + 2 = 11.6$ miles

c i The correlation is unchanged by a linear transformation on the data. It remains 0.84

ii The equation of the regression line becomes

$$\begin{aligned} Y &= 1.6y \\ &= 1.6(1.2x + 2) \\ &= 1.6\left(\frac{1.2}{1.6}X + 2\right) \\ &= 1.2X + 3.2 \end{aligned}$$

Since both axes undergo the same linear transformation, the gradient remains unchanged but the intercept is adjusted (what was an intercept of 2 miles is now 3.2 km).

7 Core: Probability

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 7A

16 a Estimate $\text{P(negative side effect)} = \frac{26}{350} = \frac{13}{175} \approx 0.0743$

b Expected number with side effects $= 900 \times \frac{13}{175} = 66.9$

17 $\text{P}(2) = \frac{2}{8} = \frac{1}{4}$

Expected number of twos after 30 rolls $= 30 \times \frac{1}{4} = 7.5$

18 $\text{P(not a diamond)} = \frac{3}{4}$

Expected number not diamonds $= 20 \times \frac{3}{4} = 15$

19 $\text{P}(1) = \frac{1}{n}$

So if $400 \times \text{P}(1) = 50$ then $n = 8$

20 $\text{P(Odd)} + \text{P(Even)} = 1$ because 'Odd' and 'Even' are complementary events.

$\text{P(Odd)} = 3\text{P(Even)}$ so $4\text{P(Even)} = 1$

$\text{P(Even)} = \frac{1}{4}, \text{P(Odd)} = \frac{3}{4}$

21 $\text{P(accident)} = \frac{124}{2000} = 0.062$

If the policy price is $\$A$, the company wants the expected payout to be $0.8\$A$ in order to make a 20% profit on policies sold.

The expected payout per policy is $0.062 \times \$15\,000 = \$930$

$0.8A = 930$ so $A = \$1162.50$

22 a $\frac{0.6}{0.4} = 1.5$

b $\frac{p}{1-p} = 4$

$p = 4 - 4p$

$5p = 4$

$p = 0.8$

23 Let r be the number of red balls.

Then there are $3r$ green balls and $12r$ blue balls, for a total of $16r$ balls.

$$\text{P(Red ball)} = \frac{r}{16r} = \frac{1}{16}$$

24 The area of the square is 1 and the area of the circle, which has radius $\frac{1}{2}$, is $\frac{\pi}{4}$

The probability of any point falling into the circle is therefore $\frac{\pi}{4}$.

The relative frequency in the simulation gives $\frac{78}{100} \approx \frac{\pi}{4}$

So $\pi \approx \frac{78}{25} = 3.12$

Exercise 7B

40 a

		First die					
		1	**2**	**3**	**4**	**5**	**6**
Second die	**1**	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

b $\text{P}(X \geqslant 20) = \frac{8}{36} = \frac{4}{18}$

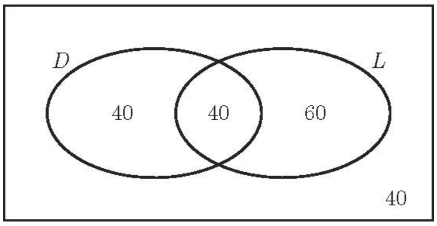
Expected number of twenties in 180 trials $= 180 \times \frac{4}{18} = 40$

41

		First die			
		1	**2**	**3**	**4**
Second die	**1**	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8
	5	6	7	8	9
	6	7	8	9	10
	7	8	9	10	11
	8	9	10	11	12

$$\text{P}(X < 10) = \frac{26}{32} = \frac{13}{16}$$

42 a



b $\mathrm{P}(D' \cap L') = \frac{40}{180} = \frac{2}{9}$

c $\mathrm{P}(L|D') = \frac{40}{80} = \frac{1}{2}$

43 $\mathrm{P}(G, Y) = \frac{12}{30} \times \frac{18}{29} = \frac{36}{145}$

$\mathrm{P}(Y, G) = \frac{18}{30} \times \frac{12}{29} = \frac{36}{145}$

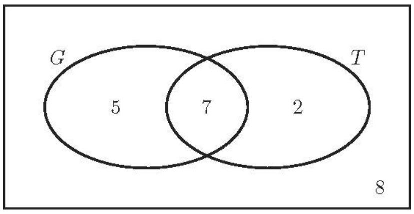
So $\mathrm{P}(\text{different colours}) = \frac{72}{145} \approx 0.496$

44 a i $\mathrm{P}(\text{late}|\text{rain}) = \frac{5}{40} = \frac{1}{8}$

ii $\mathrm{P}(\text{late}|\text{not rain}) = \frac{6}{48} = \frac{1}{8}$

b Since $\mathrm{P}(\text{late}|\text{rain}) = \mathrm{P}(\text{late}|\text{not rain})$, rain and late are independent events.

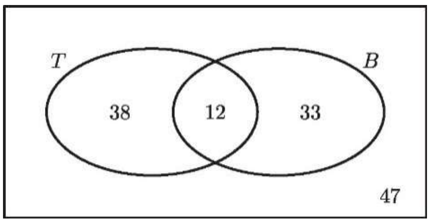
45 a



b $\mathrm{P}(G) = \frac{7}{22}$

c $\mathrm{P}(T \mid G) = \frac{5}{7}$

46



a $130 - (50 + 45 - 12) = 47$

b $\frac{83}{130}$

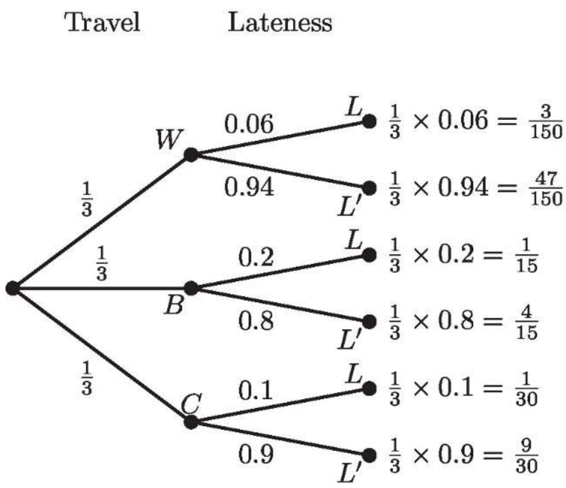
c $\frac{33}{38+3} = \frac{33}{71}$

47 There are 4 teams higher in the league and 13 teams lower in the league.

$$\mathrm{P(win)} = \frac{4}{17} \times 20\% + \frac{13}{17} \times 70\% = 58.2\%$$

48 $\mathrm{P(working)} = \frac{1}{2} \times 0.8 + \frac{1}{2} \times 0.7 = 0.75$

49



a $\mathrm{P(B \cap L)} = \frac{1}{15}$

b $\mathrm{P(L')} = \frac{47}{150} + \frac{4}{15} + \frac{9}{30} = \frac{132}{150} = \frac{22}{25} = 0.88$

50 There are $6^3 = 216$ possible throws (using ordered dice)

Means of throwing a total of 5:

1,1,3	1,2,2	1,3,1	2,1,2	2,2,1	3,1,1

Total of 6 ways, so the probability of a score 5 is $\frac{6}{216} = \frac{1}{36}$

51 a i $\mathrm{P(blue)} = \frac{77}{77+59+51} = \frac{77}{187} = \frac{7}{17}$

ii $\mathrm{P(blond)} = \frac{79}{187}$

iii $\mathrm{P(blue \cap blond)} = \frac{26}{187}$

52

$$\begin{aligned} \mathrm{P}(B) &= \mathrm{P}(A \cup B) - \big(\mathrm{P}(A) - \mathrm{P}(A \cap B)\big) \\ &= 0.9 - (0.6 - 0.2) \\ &= 0.5 \end{aligned}$$

53

$$\begin{aligned} \mathrm{P}(A \cap B) &= \big(\mathrm{P}(A) + \mathrm{P}(B)\big) - \mathrm{P}(A \cup B) \\ &= (0.7 + 0.7) - 0.9 \\ &= 0.5 \end{aligned}$$

54 a

$$\begin{aligned}\mathrm{P}(A \cap B) &= \mathrm{P}(A) \times \mathrm{P}(B|A)\\ &= \frac{2}{5} \times \frac{1}{2}\\ &= \frac{1}{5}\end{aligned}$$

b

$$\begin{aligned}\mathrm{P}(B) &= \mathrm{P}(A \cup B) - \left(\mathrm{P}(A) - \mathrm{P}(A \cap B)\right)\\ &= \frac{3}{4} - \left(\frac{2}{5} - \frac{1}{5}\right)\\ &= \frac{11}{20}\end{aligned}$$

55 a $\mathrm{P}(R_2|R_1) = \frac{39}{69} = \frac{13}{23}$

b

$$\begin{aligned}\mathrm{P}(R_1 \cap R_2) &= \frac{40}{70} \times \frac{39}{69}\\ &= \frac{52}{161}\end{aligned}$$

c

$$\begin{aligned}\mathrm{P}(G_1 \cap G_2) &= \frac{30}{70} \times \frac{29}{69}\\ &= \frac{29}{161}\end{aligned}$$

Then the probability of same colours is $\frac{52+29}{161} = \frac{81}{161} > \frac{1}{2}$

It is more likely to get the same colours than to get different colours.

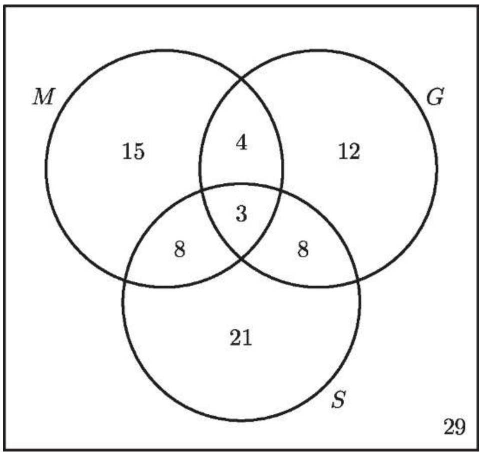
56 a Let R_n be the event of rain on day n

$$\begin{aligned}\mathrm{P}(R_1 \cup R_2) &= 1 - \mathrm{P}(R_1' \cap R_2')\\ &= 1 - (0.88^2)\\ &= 0.2256\end{aligned}$$

b

$$\begin{aligned}\mathrm{P}(R_1 \cap R_2 \cap R_3) &= 0.12^3\\ &= 0.00173\end{aligned}$$

57 **a**

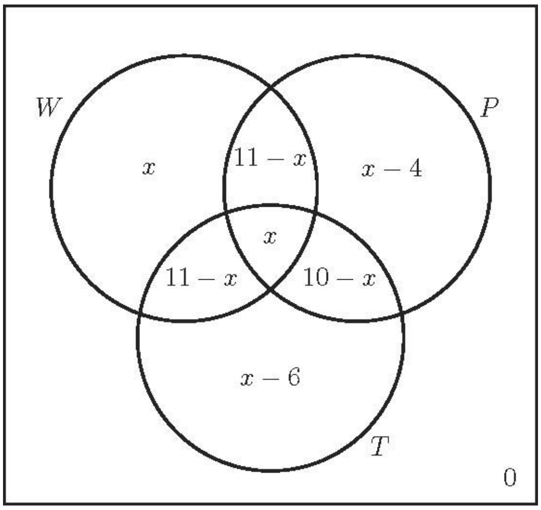


b $\mathrm{P}(S \cap G' \cap M') = \frac{21}{100} = 0.21$

c

$$\mathrm{P}(M|S) = \frac{n(M \cap S)}{n(S)} = \frac{11}{40}$$

58 **a**



b The total in all the regions of the diagram must equal 30.

$$x + (11 - x) + (11 - x) + x + (x - 4) + (10 - x) + (x - 6) = 30$$
$$x + 22 = 30$$

c $x = 8$

$$\mathrm{P}(W \cap P \cap T) = \frac{8}{30} = \frac{4}{15}$$

d

$$\mathrm{P}(P|T') = \frac{n(P \cap T')}{n(T')} = \frac{7}{15}$$

e

$$\mathrm{P}(W \cap T|2 \text{ items}) = \frac{n(W \cap T \cap P')}{n(2 \text{ items})} = \frac{3}{8}$$

59 a There are 3 options for older and younger: (B, G), (G, B) and (B, B). In the absence of other information, each is equally likely.

$$\mathrm{P}(B, B) = \frac{1}{3}$$

b There are now only two options for older and younger: BG or BB. In the absence of other information, each is equally likely.

$$\mathrm{P}(B, B) = \frac{1}{2}$$

Mixed Practice

1 $650 \times \frac{128}{200} = 416$ participants

2

	boy	girl	total
apples	16	21	37
bananas	32	14	46
strawberries	11	21	32
total	59	56	115

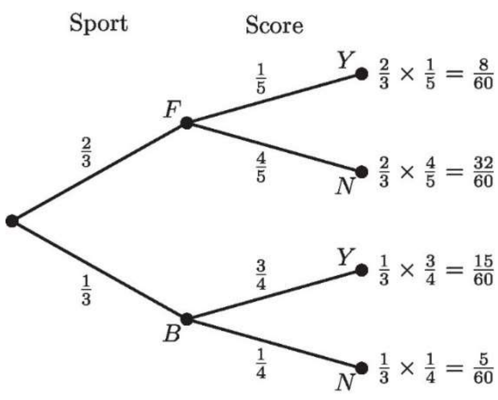
a $\mathrm{P}(\text{girl}) = \frac{56}{115}$

b $\mathrm{P}(\text{girl} \cap \text{apples}) = \frac{21}{115}$

c $\mathrm{P}(\text{banana}|\text{boy}) = \frac{32}{59}$

d $\mathrm{P}(\text{girl}|\text{strawberry}) = \frac{21}{32}$

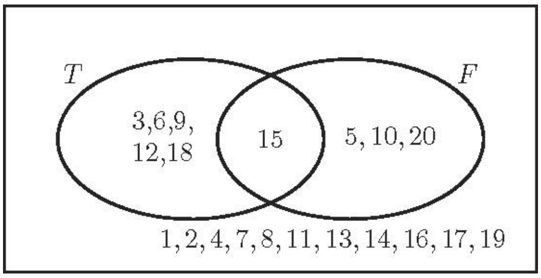
3



a $\mathrm{P}(F \cap Y) = \frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$

b $\mathrm{P}(N) = \frac{32}{60} + \frac{5}{60} = \frac{37}{60}$

4 **a**



b **i** $\mathrm{P}(F) = \frac{4}{20} = \frac{1}{5}$

ii $\mathrm{P}(F|T') = \frac{3}{14}$

5 **a**

$$\begin{aligned}\mathrm{P}(F \cap G) &= \mathrm{P}(F) + P(G) - P(F \cup G)\\ &= 0.4 + 0.6 - (1 - 0.2)\\ &= 0.2\end{aligned}$$

b

$$\begin{aligned}\mathrm{P}(G|F') &= \frac{\mathrm{P}(G \cap F')}{\mathrm{P}(F')}\\ &= \frac{0.4}{0.6}\\ &= \frac{2}{3}\end{aligned}$$

6 **a**

$$\begin{aligned}\mathrm{P}(A \cap B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B)\\ &= 0.6 + 0.3 - 0.72\\ &= 0.18\end{aligned}$$

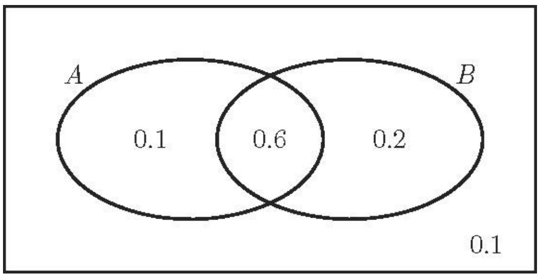
b Since $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$, A and B are independent events.

7 $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

If A and B are independent then $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$

$$\begin{aligned}\text{So } \mathrm{P}(A \cup B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A) \times \mathrm{P}(B)\\ &= 0.6 + 0.8 - 0.6 \times 0.8\\ &= 1.4 - 0.48\\ &= 0.92\end{aligned}$$

8 **a**



b $\mathrm{P}(A|B) = \frac{0.6}{0.8} = \frac{3}{4} = 0.75$

c $\mathrm{P}(B|A') = \frac{0.2}{0.3} = \frac{2}{3} = 0.667$

9 **a** **i**

$$\begin{aligned}\mathrm{P}(1) &= \mathrm{P}(1|H) \times \mathrm{P}(H) + \mathrm{P}(1|T) \times \mathrm{P}(T)\\ &= \frac{1}{6} \times \frac{1}{3} + \frac{1}{4} \times \frac{2}{3}\\ &= \frac{1}{18} + \frac{3}{18}\\ &= \frac{2}{9}\end{aligned}$$

ii

$$\begin{aligned}\mathrm{P}(6) &= \mathrm{P}(6|H) \times \mathrm{P}(H) + \mathrm{P}(6|T) \times \mathrm{P}(T)\\ &= \frac{1}{6} \times \frac{1}{3} + 0 \times \frac{2}{3}\\ &= \frac{1}{18}\end{aligned}$$

b

$$\begin{aligned}\mathrm{P}(3 \cup 6) &= \mathrm{P}(3 \cup 6|H) \times \mathrm{P}(H) + \mathrm{P}(3 \cup 6|T) \times \mathrm{P}(T)\\ &= \frac{1}{3} \times \frac{1}{3} + \frac{1}{4} \times \frac{2}{3}\\ &= \frac{2}{18} + \frac{3}{18}\\ &= \frac{5}{18}\end{aligned}$$

10 Let S be the event that the number is a multiple of 7 and N be the event that the number is a multiple of 9.

a $\frac{1000}{7} = 142.9$

$$\begin{aligned}\mathrm{P}(S) &= \frac{143}{1000}\\ &= 0.143\end{aligned}$$

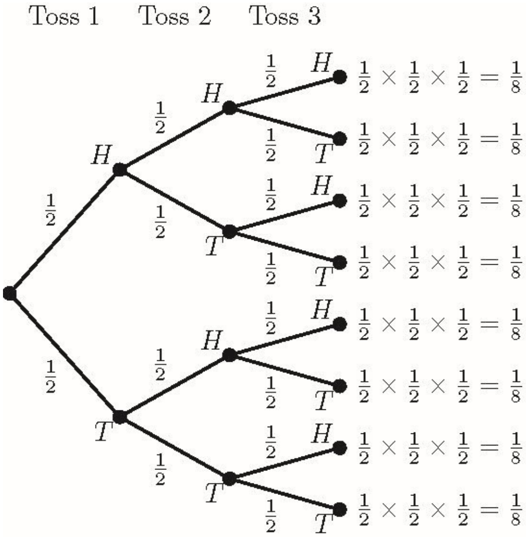
b $\frac{1000}{9} = 111.19$

$$\begin{aligned}\mathrm{P}(N) &= \frac{111}{1000}\\ &= 0.111\end{aligned}$$

c $\frac{1000}{63} = 15.9$

$$\begin{aligned}\mathrm{P}(S \cup N) &= \mathrm{P}(S) + \mathrm{P}(N) - \mathrm{P}(S \cap N)\\ &= 0.142 + 0.111 - 0.015\\ &= 0.238\end{aligned}$$

11 a



b $P(T,T,T)=\frac{1}{8}$

c $P(\text{at least one } H)=1-P(T,T,T)=\frac{7}{8}$

d $P(2H+1T)=P(H,H,T)+P(H,T,H)+P(T,H,H)=\frac{3}{8}$

12 For Asher:

$$P(WB\cup BW)=\frac{8}{14}\times\frac{6}{13}+\frac{6}{14}\times\frac{8}{13}$$
$$=\frac{48}{91}$$

For Elsa:

$$P(WB\cup BW)=\frac{8}{14}\times\frac{6}{14}+\frac{6}{14}\times\frac{8}{14}$$
$$=\frac{48}{98}=\frac{24}{49}$$

Asher has a higher chance of selecting one of each colour.

13

$$P(RR\cup WW\cup BB)=\frac{8}{19}\times\frac{7}{18}+\frac{6}{19}\times\frac{5}{18}+\frac{5}{19}\times\frac{4}{18}$$
$$=\frac{106}{342}$$
$$=\frac{53}{171}$$
$$\approx 0.310$$

14 a

$$P(R_2|B_1)=\frac{8}{23}$$

b

$$\begin{aligned} P(R_1 \cup R_2) &= P(R_1) + P(R_2) - P(R_1 \cap R_2) \\ &= \frac{8}{24} + \frac{8}{24} - \frac{8}{24} \times \frac{7}{23} \\ &= \frac{13}{23} \end{aligned}$$

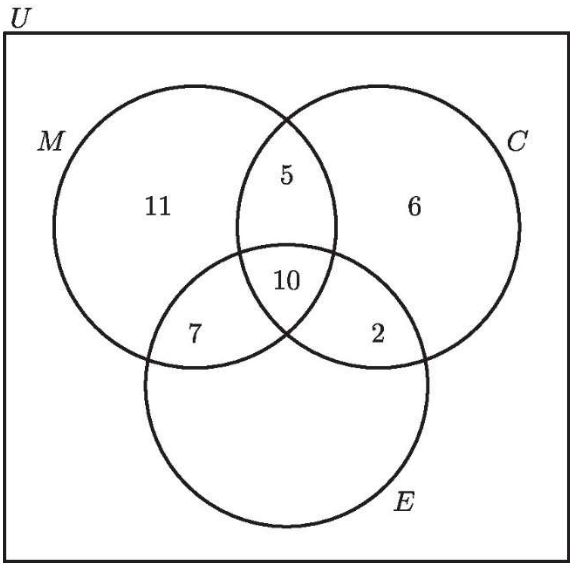
15 a $P(R) = \frac{3}{12} = \frac{1}{4}$

b $P(G_1B_2) = \frac{2}{12} \times \frac{7}{11} = \frac{7}{66}$

c

$$\begin{aligned} P(G_1G_2 \cup R_1R_2 \cup B_1B_2) &= \frac{2}{12} \times \frac{1}{11} + \frac{3}{12} \times \frac{2}{11} + \frac{7}{12} \times \frac{6}{11} \\ &= \frac{25}{66} \end{aligned}$$

16 a



b 16

c i $22 - 19 = 3$

ii Total in the diagram studying at least one of the subjects:
$11 + 5 + 6 + 7 + 10 + 2 + 3 = 44$

Total who study none of these subjects: $100 - 44 = 56$

d i $P(E) = \frac{22}{100} = \frac{11}{50} = 0.22$

ii $P(M \cap C \cap E') = \frac{5}{100} = \frac{1}{20} = 0.05$

iii $P(M' \cap E') = \frac{62}{100} = \frac{31}{50} = 0.62$

iv

$$\mathrm{P}(M'|E') = \frac{\mathrm{P}(M' \cap E')}{\mathrm{P}(E')} = \frac{0.62}{0.78} = \frac{31}{39}$$

17 $\mathrm{P}(R_1R_2) = \frac{10}{10+n} \times \frac{9}{9+n}$

$$\mathrm{P}(Y_1Y_2) = \frac{n}{10+n} \times \frac{n-1}{9+n}$$

$$\mathrm{P(Same\ colour)} = \mathrm{P}(R_1R_2) + \mathrm{P}(Y_1Y_2) = \frac{90+n^2-n}{(10+n)(9+n)} = \frac{1}{2}$$

Rearranging:

$$2(90+n^2-n) = (10+n)(9+n)$$
$$180+2n^2-2n = 90+19n+n^2$$
$$n^2-21n+90 = 0$$

b Factorising:

$$(n-15)(n-6) = 0$$

$n = 15$ or $n = 6$

18 a $\mathrm{P}(F \cup S) = 100\%$ since all students have to learn at least one of the languages.

$$\mathrm{P}(F \cap S) = \mathrm{P}(F) + \mathrm{P}(S) - \mathrm{P}(F \cup S) = 40\% + 75\% - 100\% = 15\%$$

b

$$\mathrm{P}(S \cap F') = \mathrm{P}(S) - \mathrm{P}(F \cap S) = 75\% - 15\% = 60\%$$

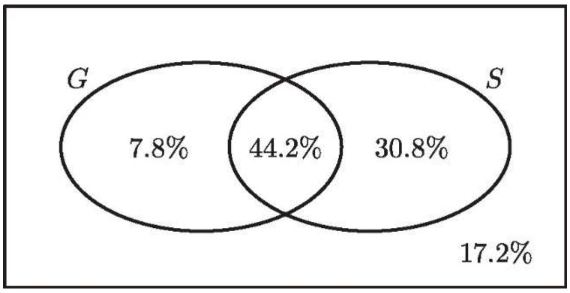
c i

$$\mathrm{P}(G \cap S) = \mathrm{P}(S|G) \times \mathrm{P}(G) = 85\% \times 52\% = 0.442 = 44.2\%$$

ii $\mathrm{P(G \cap S)} = 44.2\%$ and $\mathrm{P}(G) \times \mathrm{P}(S) = 52\% \times 75\% = 39\%$

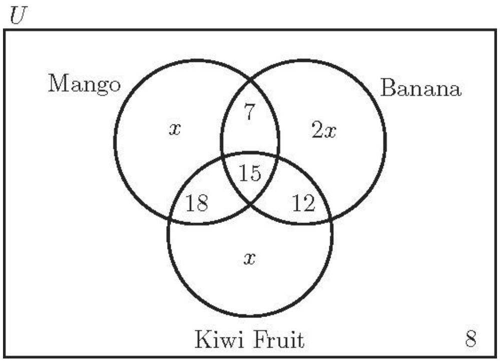
Since $\mathrm{P}(G \cap S) \neq \mathrm{P}(G) \times \mathrm{P}(S)$, the two events are not independent.

d



$$\begin{aligned}P(S|G') &= \frac{P(S \cap G')}{P(G')}\\ &= \frac{30.8\%}{48\%}\\ &= 0.642 = 64.2\%\end{aligned}$$

19 a and b



c $x + 7 + 2x + 18 + 15 + 12 + x + 8 = 100$

$$\begin{aligned}4x + 60 &= 100\\ x &= 10\end{aligned}$$

d **i** $n(\text{Mango}) = 50$

ii $n(\text{Mango} \cup \text{Banana}) = 82$

e **i** $P(M' \cap B' \cap K') = \frac{8}{100} = \frac{2}{25} = 0.08$

ii $P(M' \cap B \cap K) + P(M \cap B' \cap K) + P(M \cap B \cap K') = \frac{37}{100} = 0.37$

iii $P(M \cap B \cap K | M \cap B) = \frac{15}{22} = 0.682$

f $P(\text{both dislike all}) = \frac{8}{100} \times \frac{7}{99} = \frac{14}{2475} = 0.566\%$

8 Core: Probability distributions

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Tip: All the values calculated here are given exactly or to 3 significant figures.

Wherever a previously calculated value has to be used in a subsequent part of the question, the calculated value has been retained in the calculator and reused. If you consistently get a slightly different answer, you may be using rounded values in subsequent calculations, and are experiencing the effects of cumulative rounding errors.

As a rule of thumb, if your final answer is to be accurate to 3 s. f., you need to keep at least 4 and preferably 5 s.f. for all preliminary calculated values, or store results in your calculator for future use in the same question.

Exercise 8A

13 a Require $\sum \mathrm{P}(X = x) = 1$

$$\begin{aligned} 0.2 + 0.1 + 0.3 + k &= 1 \\ k &= 0.4 \end{aligned}$$

b $\mathrm{P}(X \geq 3) = 0.3 + k = 0.7$

c

$$\begin{aligned} \mathrm{E}(X) &= \sum x\,\mathrm{P}(X = x) \\ &= (1 \times 0.2) + (2 \times 0.1) + (3 \times 0.3) + 4k \\ &= 2.9 \end{aligned}$$

14 a Require $\sum \mathrm{P}(Y = y) = 1$

$$\begin{aligned} 0.1 + 0.3 + k + 2k &= 1 \\ 3k &= 0.6 \\ k &= 0.2 \end{aligned}$$

b $\mathrm{P}(Y < 6) = 0.1 + 0.3 = 0.4$

c

$$\begin{aligned} \mathrm{E}(Y) &= \sum y\,\mathrm{P}(Y = y) \\ &= (1 \times 0.1) + (3 \times 0.3) + 6k + 10(2k) \\ &= 6.2 \end{aligned}$$

15 a $\mathrm{P}(\mathrm{R}, \mathrm{R}) = \frac{8}{14} \times \frac{7}{13} = \frac{4}{13}$

$\mathrm{P}(\mathrm{R}, \mathrm{Y}) = \frac{8}{14} \times \frac{6}{13} = \frac{24}{91}$

$P(Y, R) = \frac{6}{14} \times \frac{8}{13} = \frac{24}{91}$

$P(Y, Y) = \frac{6}{14} \times \frac{5}{13} = \frac{15}{91}$

x	0	1	2
$P(X = x)$	$\frac{4}{13}$	$\frac{48}{91}$	$\frac{15}{91}$

b

$$E(X) = \sum x\, P(X = x)$$
$$= 0 + \left(1 \times \frac{48}{91}\right) + \left(2 \times \frac{15}{91}\right)$$
$$= \frac{78}{91} \approx 0.857$$

16 a $P(H, H) = \frac{13}{52} \times \frac{12}{51} = \frac{1}{17}$

b $P(H, H') = \frac{13}{52} \times \frac{39}{51} = \frac{13}{68}$

$P(H', H) = \frac{39}{52} \times \frac{13}{51} = \frac{13}{68}$

$P(H', H') = \frac{39}{52} \times \frac{38}{51} = \frac{19}{34}$

h	0	1	2
$P(H = h)$	$\frac{19}{34}$	$\frac{13}{34}$	$\frac{1}{17}$

c

$$E(H) = \sum h\, P(H = h)$$
$$= 0 + \left(1 \times \frac{13}{34}\right) + \left(2 \times \frac{1}{17}\right)$$
$$= \frac{17}{34}$$
$$= 0.5$$

17 Two outcomes:

Heads: Olivia has a loss of £2

Tails: Olivia has a gain of £3

Olivia's expected gain is $\frac{1}{2} \times (-£2) + \frac{1}{2} \times £3 = 50$ pence

Since Olivia has an expectation greater than zero, the game is biased in her favour, so is not fair.

18 Two outcomes:

Diamond: Shinji has a loss of £3

Not diamonds: Shinji has a gain of £n

Shinji's expected gain is $\frac{1}{4} \times (-£3) + \frac{3}{4} \times £n = (75n - 75)$ pence

The game is 'fair' (Shinji's expected gain is zero) if $n = 1$

(This is a zero-sum game, so if fair for Shinji then it is also fair for Maria)

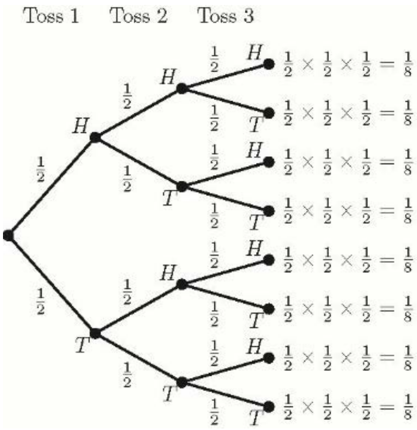
19 Let X be the number or tails and therefore the number of dollars paid out.

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\begin{aligned}E(X) &= \sum x\,P(X = x)\\ &= 0 + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right)\\ &= \frac{12}{8}\\ &= 1.5\end{aligned}$$

So the stall should charge \$1.50 to make the game fair.

20 **a**



b

Let X be the number of heads.

$$P(X = 2) = \frac{3}{8}$$

c

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

d

$$\begin{aligned}E(X) &= \sum x\,P(X = x)\\ &= 0 + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right)\\ &= \frac{12}{8}\\ &= 1.5\end{aligned}$$

21 a

		First die			
		1	**2**	**3**	**4**
Second Die	**1**	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

x	2	3	4	5	6	7	8
$\mathrm{P}(X = x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

d

$$\begin{aligned} \mathrm{E}(X) &= \sum x\,\mathrm{P}(X = x) \\ &= \left(2 \times \frac{1}{16}\right) + \left(3 \times \frac{1}{8}\right) + \left(4 \times \frac{3}{16}\right) + \left(5 \times \frac{1}{4}\right) + \left(6 \times \frac{3}{16}\right) + \left(7 \times \frac{1}{8}\right) + \left(8 \times \frac{1}{16}\right) \\ &= 5 \end{aligned}$$

22 a $\mathrm{P}(X = 3) = \frac{1}{15}(6) = \frac{2}{5} = 0.4$

b

$$\begin{aligned} \mathrm{E}(X) &= \sum x\,\mathrm{P}(X = x) \\ &= \left(1 \times \frac{4}{15}\right) + \left(2 \times \frac{5}{15}\right) + \left(3 \times \frac{6}{15}\right) \\ &= \frac{32}{15} = 2.13 \end{aligned}$$

23 a Require $\sum \mathrm{P}(y = y) = 1$

$$\begin{aligned} k(3 + 4 + 5) &= 1 \\ k &= \frac{1}{12} \end{aligned}$$

b

$$\begin{aligned} \mathrm{P}(Y \geq 5) &= \mathrm{P}(Y = 5) + \mathrm{P}(Y = 6) \\ &= 4k + 5k \\ &= \frac{3}{4} = 0.75 \end{aligned}$$

c

$$\begin{aligned} \mathrm{E}(Y) &= \sum y\,\mathrm{P}(Y = y) \\ &= \left(4 \times \frac{3}{12}\right) + \left(5 \times \frac{4}{12}\right) + \left(6 \times \frac{5}{12}\right) \\ &= \frac{62}{12} = 5.17 \end{aligned}$$

24 a Require $\sum \mathrm{P}(X=x)=1$

$$c\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=1$$
$$\frac{25}{12}c=1$$
$$c=\frac{12}{25}=0.48$$

b

x	1	2	3	4
$\mathrm{P}(X=x)$	$\frac{12}{25}$	$\frac{12}{50}$	$\frac{12}{75}$	$\frac{12}{100}$

$$\mathrm{P}(X\le 3)=\frac{88}{100}$$

So $\mathrm{P}(X=2|X\le 3)=\frac{\left(\frac{12}{50}\right)}{\left(\frac{88}{100}\right)}=\frac{24}{88}=\frac{3}{11}$

c

$$\begin{aligned}\mathrm{E}(X)&=\sum x\,\mathrm{P}(X=x)\\&=\left(1\times\frac{12}{25}\right)+\left(2\times\frac{12}{50}\right)+\left(3\times\frac{12}{75}\right)+\left(4\times\frac{12}{100}\right)\\&=1.92\end{aligned}$$

25 Require $\sum \mathrm{P}(X=x)=1$

$a+0.2+0.3+b=1$

$a+b=0.5$ (1)

$$\begin{aligned}\mathrm{E}(X)&=\sum x\,\mathrm{P}(X=x)\\&=(1\times a)+(2\times 0.2)+(3\times 0.3)+(4\times b)\\&=a+4b+1.3\\&=2.4\end{aligned}$$

$a+4b=1.1$ (2)

$(2)-(1): 3b=0.6$

So $b=0.2, a=0.3$

Exercise 8B

19 a $X\sim \mathrm{B}\left(10,\frac{1}{6}\right)$

b From GDC: $\mathrm{P}(X=2)=0.291$

c From GDC:

$$\begin{aligned}\mathrm{P}(X>2)&=1-\mathrm{P}(X\leqslant 2)\\&=1-0.775\\&=0.225\end{aligned}$$

20 **a** Let X be the number of times scored in 12 shots.

$X \sim (12, 0.85)$

From GDC: $\mathrm{P}(X = 10) = 0.292$

b From GDC

$$\begin{aligned}\mathrm{P}(X \geqslant 10) &= 1 - P(X \leqslant 9)\\ &= 1 - 0.264\\ &= 0.736\end{aligned}$$

c $\mathrm{E}(X) = 12 \times 0.85 = 10.2$

21 **a** From GDC: $\mathrm{P}(Y = 11) = 0.160$

b From GDC

$$\begin{aligned}\mathrm{P}(X > 9) &= 1 - \mathrm{P}(X \leqslant 9)\\ &= 1 - 0.128\\ &= 0.872\end{aligned}$$

c From GDC

$$\begin{aligned}\mathrm{P}(7 \leqslant X < 10) &= \mathrm{P}(X \leqslant 9) - \mathrm{P}(X \leqslant 6)\\ &= 0.128 - 0.006\\ &= 0.121\end{aligned}$$

d $\mathrm{Var}(X) = 20 \times 0.6 \times 0.4 = 4.8$

22 Let X be the number of correct answers. $X \sim \mathrm{B}(25,0.2)$

a

$$\begin{aligned}\mathrm{P}(X < 10) &= \mathrm{P}(X \leqslant 9)\\ &= 0.983\end{aligned}$$

b $\mathrm{E}(X) = 25 \times 0.2 = 5$

c

$$\begin{aligned}\mathrm{P}(X > 5) &= 1 - \mathrm{P}(X \leqslant 5)\\ &= 1 - 0.617\\ &= 0.383\end{aligned}$$

23 **a** Assume independent events and constant probability:

Each employee gets a cold with the same probability.

One employee having a cold has no impact on the probability of another employee catching a cold.

b For an infectious condition, independence is questionable; if one employee has a cold, one would imagine the probability of others getting a cold would be significantly increased.

c Let X be the number of employees suffering from a cold. $X \sim \mathrm{B}(80,0.012)$.

From GDC: $\mathrm{P}(X = 3) = 0.0560$

d

$$\begin{aligned}\mathrm{P}(X > 3) &= 1 - \mathrm{P}(X \leqslant 3)\\ &= 1 - 0.9841\\ &= 0.0159\end{aligned}$$

24 Let X be the number of sixes rolled in 20 rolls. $X \sim \text{B}\left(20, \frac{1}{6}\right)$.

$$\text{E}(X) = \frac{20}{6} = 3.33$$

$$\begin{aligned} \text{P}(X > 3.33) &= 1 - \text{P}(X \leqslant 3) \\ &= 1 - 0.567 \\ &= 0.433 \end{aligned}$$

25 a Let X be the number of eggs broken in one box. $X \sim \text{B}(6, 0.06)$.

$$\begin{aligned} \text{P}(X \geqslant 1) &= 1 - \text{P}(X = 0) \\ &= 1 - 0.690 \\ &= 0.310 \end{aligned}$$

b Let Y be the number of boxes out of ten which will be returned. $Y \sim \text{B}(10, 0.310)$.

Tip: Always clearly define your variables and use different letters for different variables. This is very important for making your working clear in questions like this where one variable has parameters calculated from the distribution of another.

$$\begin{aligned} \text{P}(Y \geqslant 1) &= 1 - \text{P}(Y = 0) \\ &= 1 - 0.0245 \\ &= 0.976 \end{aligned}$$

c

$$\begin{aligned} \text{P}(Y > 2) &= 1 - \text{P}(Y \leqslant 2) \\ &= 1 - 0.3565\ldots \\ &= 0.643 \end{aligned}$$

26 Let X be the number of hits out of 10 shots. $X \sim \text{B}(10, 0.7)$.

Assume constant probability and independence between shots.

a

$$\begin{aligned} \text{P}(X \geqslant 7) &= 1 - \text{P}(X \leqslant 6) \\ &= 1 - 0.350 \\ &= 0.650 \end{aligned}$$

b Let Y be the number of rounds in which the archer hits at least seven times. $Y \sim \text{B}(5, 0.650)$.

$$\begin{aligned} \text{P}(Y \geqslant 3) &= 1 - \text{P}(Y \leqslant 2) \\ &= 1 - 0.235 \\ &= 0.765 \end{aligned}$$

27 Let X be the number of sixes rolled in ten rolls. $X \sim \text{B}(10, p)$.

Empirical data suggests that $\text{E}(X) = 2.7 = 10p$ so estimate $p = 0.27$

$X \sim \text{B}(10, 0.27)$

From GDC:

$$\begin{aligned} \text{P}(X > 4) &= 1 - \text{P}(X \leq 4) \\ &= 1 - 0.896 \\ &= 0.104 \end{aligned}$$

28 $\text{E}(X) = np = 36 \qquad (1)$

$\text{Var}(X) = np(1-p) = 3^2 = 9 \qquad (2)$

$(2) \div (1): (1-p) = \frac{9}{36} = \frac{1}{4}$

Then $p = \frac{3}{4}$ and $n = 48$

$X \sim \text{B}(48, 0.75)$

From GDC: $\text{P}(X = 36) = 0.132$

29 **a** Let X be the number of defective components in a pack of ten. $X \sim \text{B}(10, 0.003)$

$\text{P}(X \geq 1) = 1 - \text{P}(X = 0)$
$= 1 - 0.9704$
$= 0.0296$

b A batch is rejected if both of two selected packs contain at least one defective component.

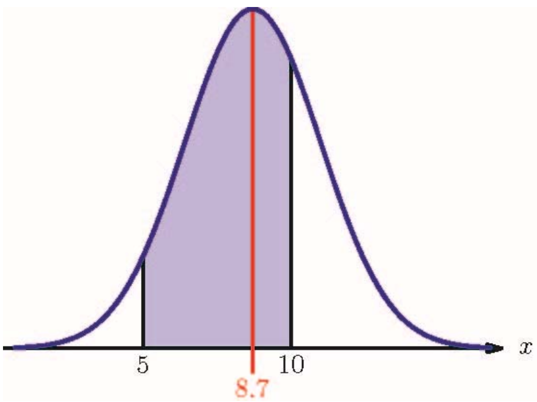
The probability of rejecting a batch is therefore $0.0296^2 = 0.000876$

c In a manufacturing situation, defective components may arise due to a faulty manufacturing machine or lower quality materials, either of which might be expected to affect multiple components.

Exercise 8C

13 No; the data suggest the distribution is not symmetrical

14 **a**



b Let X be tree height in metres. $X \sim \text{N}(8.7, 2.3^2)$.

From GDC: $\text{P}(5 < X < 10) = 0.660$

15 Let X be phone battery life in hours. $X \sim \text{N}(56, 8^2)$.

a From GDC: $\text{P}(50 < X < 60) = 0.465$

b From GDC: $\text{P}(X > 72) = 0.0228$

16 Let X be tree height in metres. $X \sim \text{N}(17.2, 6.3^2)$.

a From GDC: $\text{P}(15 < X < 20) = 0.308$

b From GDC: $\text{P}(X > 20) = 0.328$

17 Let X be the time Charlotte runs a 400 m race, in seconds. $X \sim N(62.3, 4.5^2)$.

a From GDC: $\mathrm{P}(X > 65) = 0.274$

b Let Y be the number of races out of 38 in which she ran over 65 seconds.

$Y \sim \mathrm{B}(38, 0.274)$
$\mathrm{E}(Y) = 38 \times 0.274 = 10.4$

c From GDC: $\mathrm{P}(X < 59.7) = 0.282$

18 Let X be the time (in minutes) taken to complete a puzzle. $X \sim \mathrm{N}(4.5, 1.5^2)$.

a From GDC: $\mathrm{P}(X > 7) = 0.0478$

b Require x such that $\mathrm{P}(X < x) = 0.9$

From GDC: $x = 6.42$ minutes

19 Let X be the time (in hours) of screen time. $X \sim \mathrm{N}(4.2, 1.3^2)$.

a From GDC: $\mathrm{P}(X > 6) = 0.0831$

b Require T such that $\mathrm{P}(X < T) = 0.95$

From GDC: $T = 6.34$ hours

c From GDC: $\mathrm{P}(X < 3) = 0.178$

Let Y be the number of teenagers in a group of 350 who get less than 3 hours of screen time per day.

$Y \sim \mathrm{B}(350, 0.178)$
$\mathrm{E}(Y) = 350 \times 0.178 = 62.3$

20 Let X be the distance (in metres) achieved in long jump for this group. $X \sim \mathrm{N}(5.2, 0.6)$.

a From GDC: $\mathrm{P}(X > 6) = 0.151$

Let Y be the number of competitors out of 30 who jump further than 6 m.

$\mathrm{E}(Y) = 30 \times 0.151 = 4.53$

b Require x such that $\mathrm{P}(X < x) = 0.95$

From GDC: $x = 6.47$ m

21 Let T be the time (in seconds) taken to run 100 m for this group. $T \sim \mathrm{N}(14.3, 2.2)$.

Require t such that $\mathrm{P}\left(t < t_q\right) = 0.15$ where t_q is the greatest possible qualifying time.

Now, when $P\left(z < Z_q\right) = 0.15$, we have that $Z_q = 1.03624$.

I.e. $Z_q = \frac{14.3 - t_q}{\sqrt{2.2}} = 1.03624$

Hence, $t_q = 12.763 = 12.8$ s

22 $X \sim \mathrm{N}(12, 5)$

a Require x such that $\mathrm{P}(X < x) = 0.75$

From GDC: $x = 15.4$ s

b Require x_U such that $\mathrm{P}(X < x_U) = 0.75$ and x_L such that $\mathrm{P}(X < x_L) = 0.25$

From GDC: $x_U = 15.4$ s and $x_L = 8.63$ s

So $\mathrm{IQR} = x_U - x_L = 6.74$ s

23 $X \sim \mathrm{N}(3.6, 1.2)$

Require x such that $\mathrm{P}(X < x) = 0.8$

From GDC: $x = 4.61$

24 $X \sim \mathrm{N}(17, 3.2^2)$

From GDC: $\mathrm{P}(X > 15) = 0.734$

Require k such that $\mathrm{P}(X > k) = 0.734 - 0.62$

From GDC: $k = 20.9$

25 $Y \sim \mathrm{N}(13.2, 5.1^2)$

From GDC: $\mathrm{P}(X < 17.3) = 0.789$

Require c such that $\mathrm{P}(X < c) = 0.789 - 0.14$

From GDC: $k = 15.2$

26 For a Normal distribution to apply, require a symmetrical unimodal distribution with approximately 95% of the distribution within 2 standard deviations from the mean.

The test data show that the minimum mark (0%) is only 1.75 standard deviations below the mean, which would mean the distribution cannot have that Normal bell-shape.

(An alternative way of looking at it would be that if the distribution were $X \sim \mathrm{N}(35, 20^2)$ then there would be a predicted probability of $\mathrm{P}(X < 0) = 0.04 = 4\%$ that a member of the population scores less than 0%.)

27 **a** The distribution appears symmetrical, with the central 50% occupying an interval equivalent to between half and two thirds of each of the 25% tails of the distribution.

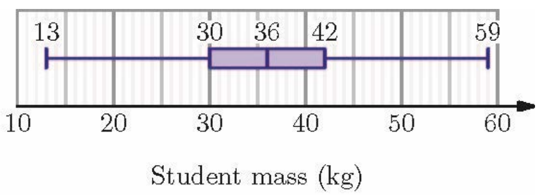
b Let X be the weights of children; $X \sim N(36, 8.5^2)$

Require x_U such that $\mathrm{P}(X < x_U) = 0.75$ and x_L such that $\mathrm{P}(X < x_L) = 0.25$

From GDC: $x_U = 41.7$ and $x_L = 30.3$

This gives $\mathrm{IQR} = 11.7$.

Given there are no outliers, we understand the minimum to be no less than $x_L - 1.5 \times \mathrm{IQR} = 12.8$ and the maximum to be no greater than $x_U + 1.5\mathrm{IQR} = 59.2$



Tip: If you are only considering the Normal model aspect of the question, you might consider that the correct upper and lower boundaries would be more like $\mu \pm 3\sigma$ but then – by the definition of outliers in a box plot – the minimum and maximum values you plot for the of the population would be outliers, which the question explicitly excludes.

28 Let X be the sprinter's reaction time, in seconds. $X \sim \mathrm{N}(0.2, 0.1^2)$.

a From GDC: $\mathrm{P}(X < 0) = 0.0228$.

If the reaction time is recorded as negative, that means the sprinter anticipated the start signal (left the blocks before the signal was given).

b From GDC: $P(X < 0.1) = 0.159$.

c Let Y be the number of races out of ten in which the sprinter gets a false start.

$$Y \sim B(10, 0.159)$$

$$\begin{aligned} P(Y > 1) &= 1 - P(Y \leqslant 1) \\ &= 1 - 0.513 \\ &= 0.488 \end{aligned}$$

d It is assumed that the sprinter's probability of a false start is constant and that false starts are independent, so that Y can be modelled by a binomial distribution. Since repeat offences might lead to disqualification or being barred from future events, it is likely that after one false start, the sprinter would adjust behaviour; this might be expected to equate to raising the mean reaction time above 0.2 s, for example, and would mean that the set of 10 races would not have a constant probability of a false start, and would not be independent of each other.

29 Let X be the mass of an egg. $X \sim N(60, 5^2)$.

$P(X < 53) = 0.0808$

$P(53 < X < 63) = 0.645$

$P(X > 63) = 0.274$

Let Y be the income from an egg, in cents

y	0	12	16
$P(Y = y)$	0.0808	0.645	0.274

$$\begin{aligned} E(Y) &= \sum y\, P(Y = y) \\ &= (0 \times 0.0808) + (12 \times 0.645) + (16 \times 0.274) \\ &= 12.1 \end{aligned}$$

So the expected value of each egg is 12.1 ¢

Then the expected value of 6000 eggs is 6000×12.1 ¢ $= \$728$ (to 3 s.f.)

Mixed Practice

1 **a** Require $\sum P(X = x) = 1$

$$\begin{aligned} 0.2 + 0.2 + 0.1 + k &= 1 \\ k &= 0.5 \end{aligned}$$

b $P(X \geqslant 3) = 0.1 + k = 0.6$

c

$$\begin{aligned} E(X) &= \sum x\, P(X = x) \\ &= (1 \times 0.2) + (2 \times 0.2) + (3 \times 0.1) + 4k \\ &= 2.9 \end{aligned}$$

2 Let X be the number of sixes rolled in twelve rolls. $X \sim B\left(12, \frac{1}{6}\right)$.

a From GDC:

$P(X = 2) = 0.296$

b From GDC:

$$\begin{aligned} P(X > 2) &= 1 - P(X \leqslant 2) \\ &= 1 - 0.677 \\ &= 0.323 \end{aligned}$$

3 Let X be film length, in minutes. $X \sim N(96, 12^2)$.

a From GDC: $P(100 < X < 120) = 0.347$

b From GDC: $P(X > 105) = 0.227$

4 Let X be a score on the test. $X \sim N$.

Require x such that $P(X < x) = 0.985$

From GDC: $x = 215$

5 Let X be the number of defective plates in a random sample of twenty. $X \sim B(20, 0.021)$

From GDC

$$\begin{aligned} P(X \geqslant 1) &= 1 - P(X = 0) \\ &= 1 - 0.654 \\ &= 0.346 \end{aligned}$$

6 a

x	\$1	−\$0.50	−\$1.50
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

b Expected value of each play for Alessia is

$$\begin{aligned} E(X) &= \sum x\, P(X = x) \\ &= \left(1 \times \frac{1}{2}\right) + \left(-0.5 \times \frac{1}{3}\right) + \left(-1.5 \times \frac{1}{6}\right) \\ &= -\frac{1}{12} \end{aligned}$$

The game is not fair; Alessia expects to lose \$$\frac{1}{12}$ on each play.

7 Let X be the profit a player makes on the game.

x	-2	0	3	$N - 3$
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$

For the game to be fair, require $E(X) = 0$

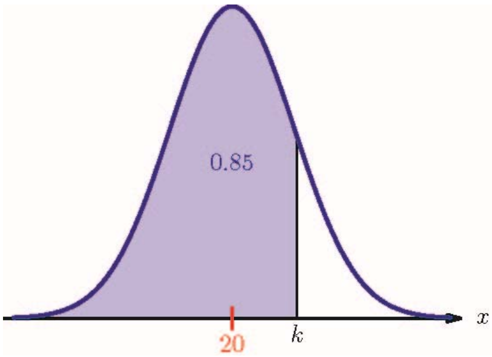
$$\begin{aligned} E(X) &= \sum x\, P(X = x) \\ &= \left(-2 \times \frac{1}{2}\right) + \left(0 \times \frac{1}{5}\right) + \left(3 \times \frac{1}{5}\right) + \frac{N-3}{10} \\ &= \frac{N-3}{10} - \frac{2}{5} \end{aligned}$$

Therefore $N - 3 = 4$ from which $N = 7$

8 $X \sim \mathrm{N}(20, 3^2)$

a From GDC: $\mathrm{P}(X \leqslant 24.5) = 0.933$

b **i**



ii From GDC: $k = 23.1$

9 $X \sim \mathrm{B}(30, p)$

a $\mathrm{E}(X) = 10 = 30P$ so $P = \frac{1}{3}$

b From GDC: $\mathrm{P}(X = 10) = 0.153$

c From GDC:

$$\begin{aligned}\mathrm{P}(X \geq 15) &= 1 - \mathrm{P}(X \leq 14)\\ &= 1 - 0.9565\\ &= 0.0435\end{aligned}$$

10 **a** Let X be the mass of an apple. $X \sim \mathrm{N}(110, 12.2^2)$.

From GDC: $\mathrm{P}(X < 100) = 0.206$

b Let Y be the number of apples in a bag of six with mass less than 100 g.

Assuming independence and constant probability from part **a**:

From GDC:

$$\begin{aligned}\mathrm{P}(Y > 15) &= 1 - \mathrm{P}(Y \leqslant 1)\\ &= 1 - 0.640\\ &= 0.360\end{aligned}$$

11 **a** Let X be the mass of an egg. $X \sim \mathrm{N}(63, 6.8^2)$.

$\mathrm{P}(X > 73) = 0.0707$

Let Y_6 be the number of very large eggs in a box of six eggs.

Assuming independence and a constant probability, $Y_6 \sim \mathrm{B}(6, 0.0707)$.

$$\begin{aligned}\mathrm{P}(Y \geqslant 1) &= 1 - \mathrm{P}(Y = 0)\\ &= 1 - 0.644\\ &= 0.356\end{aligned}$$

b Let Y_{12} be the number of very large eggs in a box of twelve. Assuming the same independence and constant probability of a very large egg: $Y_{12} \sim \mathrm{B}(12, 0.0707)$.

From GDC: $\mathrm{P}(Y_6 = 1) = 0.294$

From GDC: $\mathrm{P}(Y_{12} = 2) = 0.158$

It is more likely that a box of six contains a single very large egg.

12 Let X be jump length in metres. $X \sim \mathrm{N}(7.35, 0.8^2)$.

a From GDC: $\mathrm{P}(X > 7.65) = 0.354$

b Let Y be the number of jumps in three attempts which exceed 7.65 m. $Y \sim \mathrm{B}(3, 0.354)$.

From GDC:

$$\begin{aligned}\mathrm{P}(Y \geqslant 1) &= 1 - \mathrm{P}(Y = 0)\\ &= 1 - 0.2695\ldots\\ &= 0.730\end{aligned}$$

13 Let X be the 100 m time of a club runner in seconds. $X \sim \mathrm{N}(15.2, 1.6^2)$

From GDC: $\mathrm{P}(X < 13.8) = 0.191$

Let Y be the number of racers (out of the other seven) who have a time less than 13.8 s.

$Y \sim \mathrm{B}(7, 0.191)$

Heidi wins the race when $Y = 0$.

From GDC: $\mathrm{P}(Y = 0) = 0.227$

14 Let X be the amount of paracetamol in a tablet, in milligrams. $X \sim \mathrm{N}(500, 80^2)$.

From GDC: $\mathrm{P}(X < 380) = 0.0668$

Let Y be the number of participants in a trial of twenty five who get a dose less than 380 mg.

$Y \sim \mathrm{B}(25, 0.0668)$

From GDC:

$$\begin{aligned}\mathrm{P}(Y > 2) &= 1 - \mathrm{P}(Y \leq 0)\\ &= 1 - 0.768\\ &= 0.232\end{aligned}$$

15 a

$$\begin{aligned}\mathrm{E}(X) &= \sum x\,\mathrm{P}(X = x)\\ &= (1 \times 0.15) + (2 \times 0.25) + (3 \times 0.08) + (4 \times 0.17) + (5 \times 0.15) + (6 \times 0.20)\\ &= 3.52\end{aligned}$$

b Rolls are independent, so

$\mathrm{P}(5,6) = 0.15 \times 0.2 = 0.03$

$\mathrm{P}(6,5) = 0.2 \times 0.15 = 0.03$

So the probability of rolling a five and a six (in either order) is 0.06.

c Let X be the number of times in ten rolls that a one is rolled. $X \sim \mathrm{B}(10, 0.15)$.

From GDC:

$$\begin{aligned}\mathrm{P}(X \geqslant 2) &= 1 - \mathrm{P}(Y \leqslant 1)\\ &= 1 - 0.544\\ &= 0.456\end{aligned}$$

16 Require $\sum P(X = x) = 1$

$$\frac{1}{3} + \frac{1}{4} + a + b = 1$$

$$b = \frac{7}{12} - a$$

Let X be the gain in a single game.

x	-2	-1	0	1
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{4}$	a	$\frac{7}{12} - a$

$$\begin{aligned} E(X) &= \sum x\, P(X = x) \\ &= \left(-2 \times \frac{1}{3}\right) + \left(-1 \times \frac{1}{4}\right) + \frac{7}{12} - a \\ &= -\frac{4}{12} - a \end{aligned}$$

Expected gain per game is $-\frac{1}{2}$

So $-\frac{1}{3} - a = -\frac{1}{2}$

$$a = \frac{1}{6}$$

17 Let X be the height of a pupil at the school in cm. $X \sim N(148, 8^2)$.

a Require x_U such that $P(X < x_U) = 0.75$ and x_L such that $P(X < x_L) = 0.25$

From GDC: $x_U = 153.4$ cm and $x_L = 142.6$ cm

So IQR $= 153.4 - 142.6 = 10.8$ cm

b Outliers are more than 1.5 IQR outside the quartile marks.

From GDC:

$P(X < x_L - 1.5 \times 10.8) = 0.00349$

By symmetry, there is the same probability of a value above $x_U + 1.5$ IQR

Total probability of outlier: 0.00698 or 0.698% of the pupils.

18 For a Normal distribution to apply, require a symmetrical unimodal distribution with approximately 95% of the distribution within 2 standard deviations from the mean.

The context shows that the minimum time (0 minutes, which is clearly unrealistic anyway) is only 2 standard deviations below the mean, which would mean the distribution cannot have that Normal bell-shape.

(An alternative way of looking at it would be that if the distribution were $X \sim N(10, 5^2)$ then there would be a predicted probability of $P(X < 0) = 0.0228$ that a member of the population finishes the test before starting it!)

19 Let X be the volume of water in a bottle (in millilitres). $X \sim \text{N}(330, 5^2)$.

Require x such that $\text{P}(X < x) = 0.025$.

From GDC: $x = 320.2$

The bottle should be labelled 320 ml.

20

		First die					
		1	**2**	**3**	**4**	**5**	**6**
Second Die	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

a $\text{P}(X = 5) = \frac{4}{36} = \frac{1}{9}$

b Let Y be the number of prizes won in eight attempts. $Y \sim \text{B}\left(8, \frac{1}{9}\right)$

$\text{P}(Y = 3) = 0.0426$

21 a Let X be the time to complete a task. $X \sim \text{N}(20, 1.25^2)$.

From GDC: $\text{P}(X < 21.8) = 0.925$

b Require x such that $\text{P}(X < x) = 0.925 - 0.3 = 0.625$

From GDC: $x = 20.4$

22 a $W \sim \text{N}(1000, 4^2)$

From GDC: $\text{P}(990 < W < 1004) = 0.835$

b Require k such that $\text{P}(W < k) = 0.95$

From GDC: $k = 1006.58$

Tip: Given the context of the question, you clearly don't want to give an answer to 3s.f. here! Quick judgement suggests that the values should be 3 s.f. detail for their difference from 1000 g

c Interval symmetrical about the distribution mean, so we require 5% above the interval and 5% below the interval. Then $a = k - 1000 = 6.58$

23 $X \sim \text{B}\left(5, \frac{1}{5}\right)$

a i $\text{E}(X) = 5 \times \frac{1}{5} = 1$

ii From GDC:

$$\begin{aligned}\text{P}(X \geq 3) &= 1 - \text{P}(X \leq 2)\\ &= 1 - 0.9421\\ &= 0.0579\end{aligned}$$

b i Require $\sum P(Y = y) = 1$

$$0.67 + 0.05 + (a + 2b) + (a - b) + (2a + b) + 0.04 = 1$$
$$4a + 2b = 0.24$$

ii

$$\begin{aligned} E(Y) &= \sum y\, P(Y = y) \\ &= (0 \times 0.67) + (1 \times 0.05) + 2(a + 2b) + 3(a - b) + 4(2a + b) \\ &\quad + (5 \times 0.04) \\ &= 0.25 + 13a + 5b \\ &= 1 \end{aligned}$$

Simultaneous equations:

$4a + 2b = 0.24$ (1)

$13a + 5b = 0.75$ (2)

$2(2) - 5(1): 6a = 0.3$

So $a = 0.05$ and $b = 0.02$

c $P(Y \geqslant 3) = (a - b) + (2a + b) + 0.04 = 0.19 > 0.0579$

Bill is more likely to pass the test.

24 Let X be the distance thrown in metres. $X \sim N(40, 5^2)$.

By symmetry, $P(X < 40) = 0.5$

From GDC: $P(40 < X < 46) = 0.385$

Then $P(X > 46) = 0.115$

Let Y be the number of points achieved in a single throw.

$\boldsymbol{y}$	0	1	4
$\mathbf{P}(\boldsymbol{Y} = \boldsymbol{y})$	0.5	0.385	0.115

a i

$$\begin{aligned} E(Y) &= \sum y\, P(Y = y) \\ &= 0 + (1 \times 0.385) + (4 \times 0.115) \\ &= 0.845 \text{ points} \end{aligned}$$

ii If she throws twice, the expected value is $2 \times 0.845 = 1.69$ points.

b It is assumed that the probabilities stay the same between throws, so that each attempt follows the same distribution and that attempts are independent. However, Josie may be expected to improve as she warms up, or perhaps get worse in the later attempts as she tires. Success may lift her ability to achieve, and failure to score may make her tense up and perform worse. At any rate, we may anticipate that subsequent throws do not follow the same normal distribution, and depend on previous throws.

25 Let X be the number of sixes rolled in n throws. $X \sim \mathrm{B}\left(n, \frac{1}{6}\right)$.

Then $\mathrm{P}(X = 0) = (1 - p)^n$

$$\left(\frac{5}{6}\right)^n = 0.194$$

From GDC: $n = 9$

26 Let X be the number of heads when a fair coin is tossed n times. $X \sim (n, 0.5)$.

Require n such that $\mathrm{P}(X = 0) < 0.001$

$\mathrm{P}(X = 0) = 0.5^n$

So $0.5^n < 0.001$

Then $2^n > 1000$

The least such n is 10, since $2^{10} = 1024$

27 Let X_{10} be the number of tails from ten tosses of the biased coin. $X_{10} \sim \mathrm{B}(10, 0.57)$.

a From GDC:

$$\begin{aligned}\mathrm{P}(X_{10} \geq 4) &= 1 - \mathrm{P}(X_{10} \leq 3)\\ &= 1 - 0.0806\\ &= 0.919\end{aligned}$$

b For the fourth tail on the tenth toss, require three tails in the first nine tosses and then a tail.

Let X_9 be the number of tails from nine tosses. $X_9 \sim \mathrm{B}(9, 0.57)$

From GDC: $\mathrm{P}(X_9 = 3) = 0.0983$

So the required probability is $0.0983 \times 0.57 = 0.0561$

28

Tip: This may not look immediately like anything you have studied so far. Try to reframe the question in terms of something you do know. Many probability problems can be considered from different perspectives, to make them solvable by known methods.

For each day of the year, the number of people with that birthday will follow a $X \sim \mathrm{B}\left(100, \frac{1}{365}\right)$ distribution, assuming independence between the people.

The probability for each day that no person has a birthday that day is $\mathrm{P}(X = 0) = 0.760$

Although in this scenario, days are not truly independent (in that, each time you find a day with no birthdays, the probability of birthdays on subsequent assessed days changes slightly) the number of possible days is large enough to disregard this issue.

The total number of days expected to have no birthdays is therefore $365 \times 0.760 = 277$

29 a Assuming the responses from invited guests to be independent:

Let X be the number of invited guests who attend when he invites 5: $X \sim \mathrm{B}(5, 0.5)$

$\boldsymbol{x}$	0	1	2	3	4	5
$\mathbf{P}(\boldsymbol{X} = \boldsymbol{x})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

Let Y be the profit made, in pounds:

$\boldsymbol{x}$	0	1	2	3	4	5
$\boldsymbol{y}$	0	50	100	150	200	100
$\mathbf{P}(\boldsymbol{Y} = \boldsymbol{y})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

$$\begin{aligned}\mathrm{E}(Y) &= \sum y\,\mathrm{P}(Y = y)\\ &= \left(0 \times \frac{1}{32}\right) + \left(50 \times \frac{5}{32}\right) + \left(100 \times \frac{10}{32}\right) + \left(150 \times \frac{10}{32}\right) + \left(200 \times \frac{5}{32}\right)\\ &\qquad + \left(100 \times \frac{1}{32}\right)\\ &= 120.3\end{aligned}$$

Expected profit from 5 invitations is £120

b For four invitations sent out:

x	0	1	2	3	4	5
y	0	50	100	150	200	100
$\mathrm{P}(Y = y)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	0

$\mathrm{E}(Y) = 100$

For six invitations sent out:

x	0	1	2	3	4	5	6
y	0	50	100	150	200	100	0
$\mathrm{P}(Y = y)$	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{1}{64}$

$\mathrm{E}(Y) = 131.25$

For seven invitations sent out:

x	0	1	2	3	4	5	6	7
y	0	50	100	150	200	100	0	-100
$\mathrm{P}(Y = y)$	$\frac{1}{128}$	$\frac{7}{128}$	$\frac{21}{128}$	$\frac{35}{128}$	$\frac{35}{128}$	$\frac{21}{128}$	$\frac{7}{128}$	$\frac{1}{128}$

$\mathrm{E}(Y) = 130.5$

On the basis solely of maximum profit for a single evening, the best number of invitations to send is 6. Inviting 8 or more people leads to an expected number of guests of 4 or more, so expected profit will decrease, since each additional attendee has a negative profit effect.

In context, getting a reputation for turning invited guests away may not – in the long term – be good for business, and might start to reduce the proportion of people who attend.

9 Core: Differentiation

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 9A

27 a $\frac{\mathrm{d}V}{\mathrm{d}t} = V + 1$

b When $V = 4$, the equation in part **a** gives $\frac{\mathrm{d}V}{\mathrm{d}t} = 5$.

28

x	$\frac{\sin x^2}{\left(\frac{\pi x}{180}\right)^2}$
10	32.33
5	55.49
1	57.29
0.1	57.30

The limit appears to be 57.3.

29

x	$\frac{\ln x}{x-1}$
0.5	1.38629
0.9	1.05361
0.99	1.00503
0.999	1.00050

The limit appears to equal 1

30 a Gradient $= \frac{\text{change in } x}{\text{change in } y} = \frac{x^2-1}{x-1} = x + 1$ for $x \neq 1$

b As the value of x tends towards 1, the gradient of the chord tends towards 2. The significance of this is that the gradient of the tangent at $x = 1$ must be 2.

31

x	$\frac{\ln x}{x-1}$
10	2.6491
100	2.9563
1000	2.9955
10000	2.9996

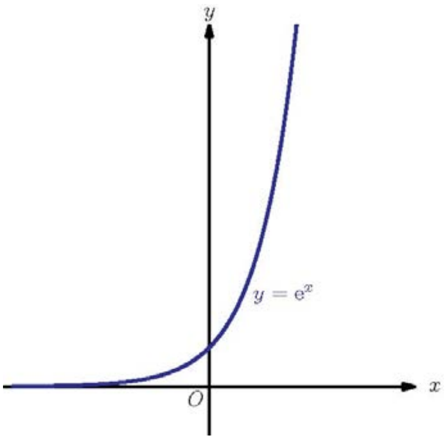
The limit appears to be 3 as x tends to ∞.

32 $\frac{dx}{dy} = 3x$ so $\frac{dy}{dx} = \frac{1}{3x}$

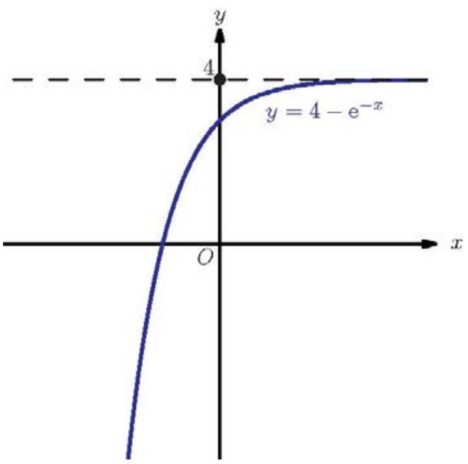
When $x = 2$, the gradient is $\frac{1}{6}$.

Exercise 9B

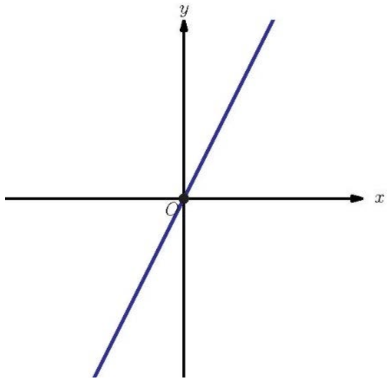
19 For example, $y = e^x$, which has derivative $\frac{dy}{dx} = e^x$



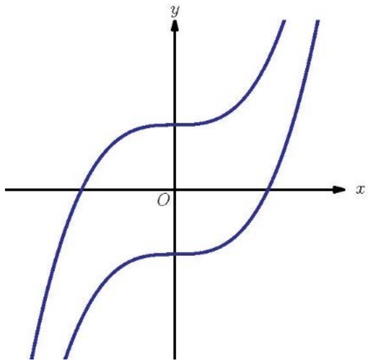
20 For example, $y = 4 - e^{-x}$, which increases towards $y = 4$ as x tends to ∞, and has derivative $\frac{dy}{dx} = e^{-x}$, which decreases towards $\frac{dy}{dx} = 0$ as x tends to ∞.



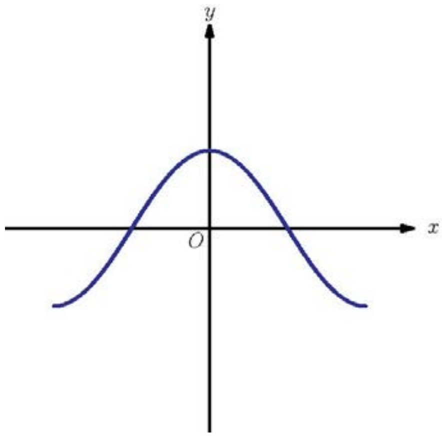
21 a The curve decreases (negative gradient) to a minimum at $x = 0$ and then rises (positive gradient) so the derivative graph should pass through the origin, from negative to positive.



b The gradient curve is non-negative throughout: it decreases to zero at $x = 0$ and then rises again, so there is a stationary point of inflexion at $x = 0$ in the graph of $f(x)$, with the rest of the graph increasing; the gradient is steeper further from $x = 0$.

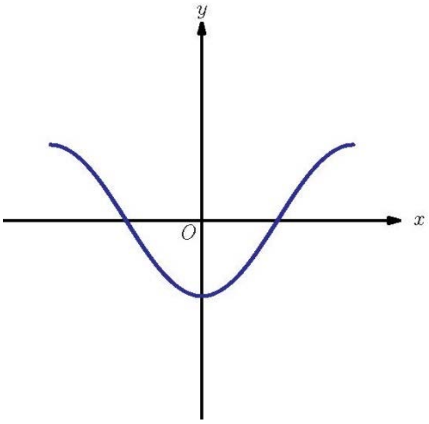


22 a The curve decreases (negative gradient) to a minimum (zero gradient), rises (positive gradient) to a maximum (zero gradient) and decreases again (negative gradient).



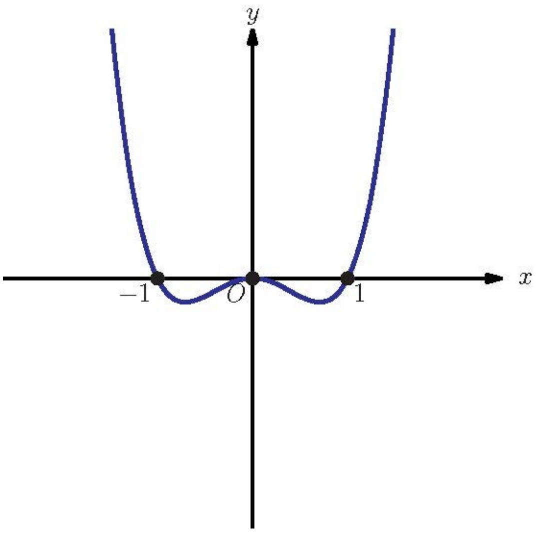
Worked solutions

b The gradient curve begins at zero, is increasingly negative then returns to zero at the origin, continues to increase and then falls back to zero, so there is a stationary point at $x = 0$ on the graph of $\mathrm{f}(x)$ and also at the two ends of the curve:



c The gradient describes the shape of a curve but not its position vertically; any vertical translation of the graph in part **b** would have the same gradient curve.

23 a $y = x^4 - x^2$



b $f(x) > 0$ for $x < -1$ and for $x > 1$

c The graph is decreasing for $x < -\frac{\sqrt{2}}{2} = -0.707$ and for $0 < x < \frac{\sqrt{2}}{2} = 0.707$

This can either be found using technology or by differentiating the curve and finding stationary points, using the methods from Section C.

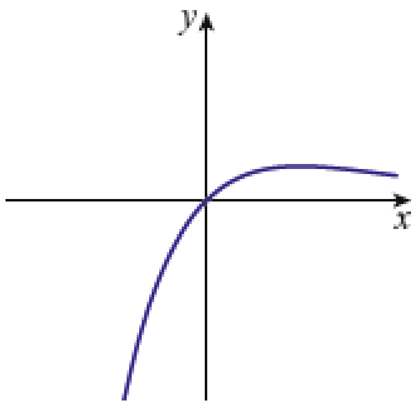
$f(x) = x^4 - x^2$ so $f'(x) = 4x^3 - 2x$

When $f'(x) = 0$, $2x(2x^2 - 1) = 0$ so turning points are at $x = 0, \pm\frac{\sqrt{2}}{2}$.

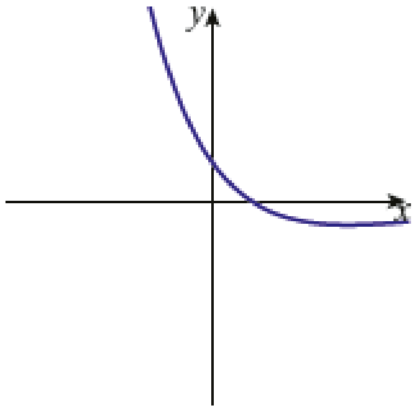
Inspecting the curve in part **a** shows clearly which regions are decreasing.

24 A increases to a stationary point at positive x then has a shallow negative gradient; its derivative graph is shown in E.

A

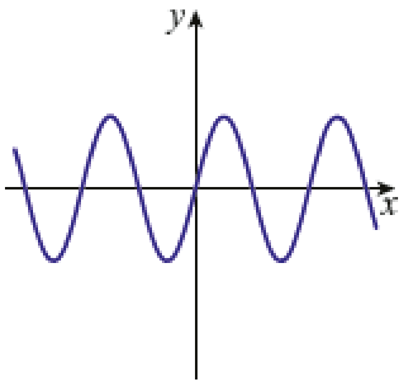


E

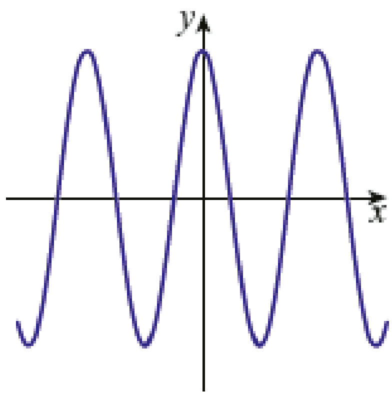


B oscillates regularly with a positive gradient at $x = 0$; its gradient is shown in D.

B

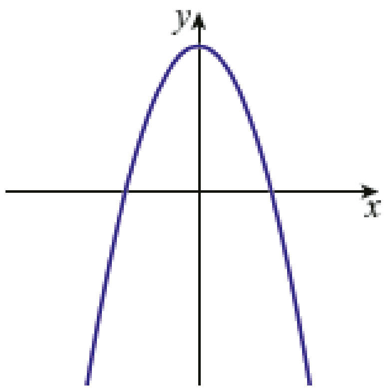


D

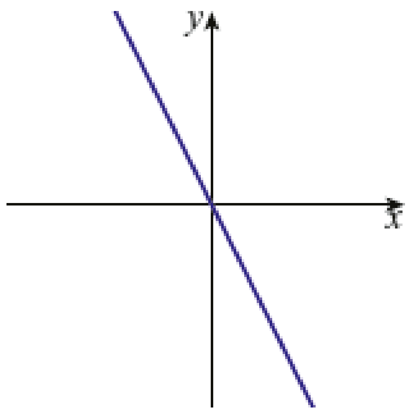


F increases to a stationary point at $x = 0$ and then decreases; its gradient is shown in C

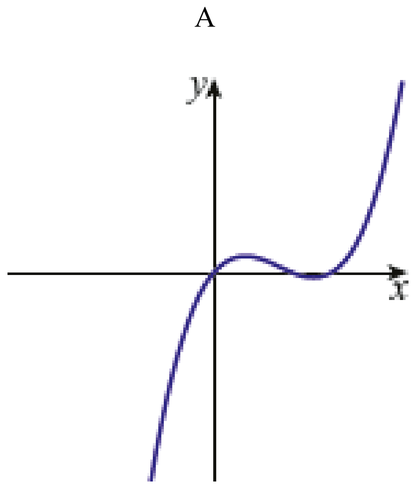
F

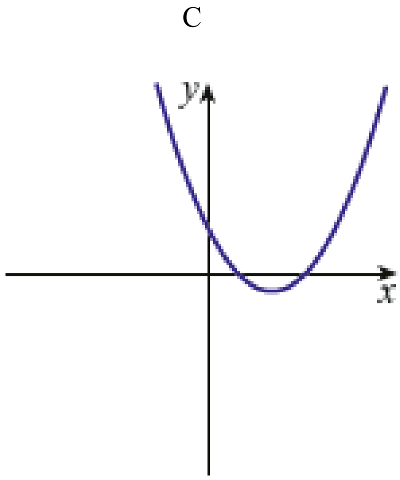


C

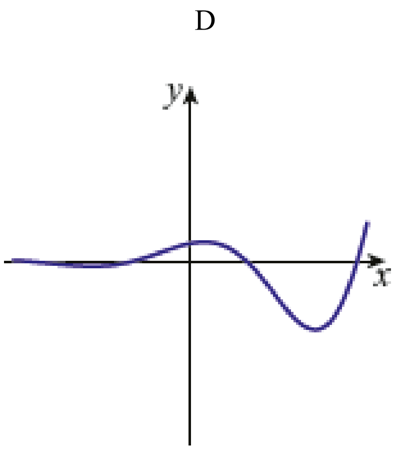


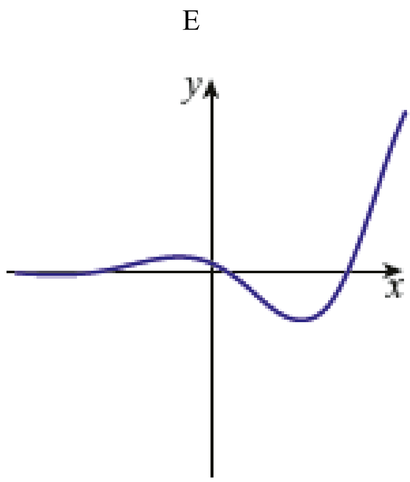
25 A increases through the origin to a maximum point for some $x > 0$, falls to a minimum and then increases with increasing gradient. Its derivative graph is given in C.



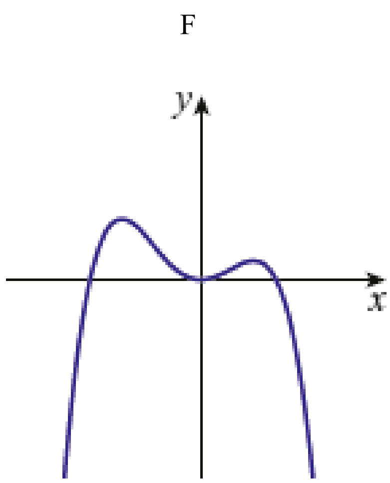


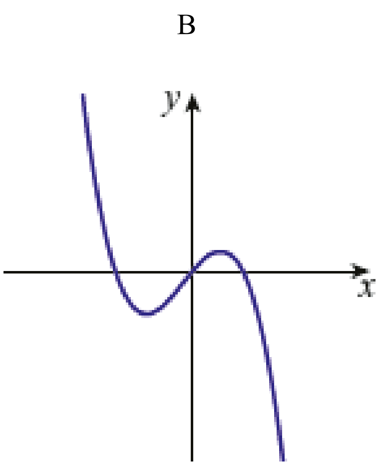
D has a minimum for some $x < 0$, then rises to a maximum at some $x > 0$ and falls to a further minimum value before rising steeply. Its derivative graph is given in E.





F increases to a maximum for some $x < 0$, falls to a minimum at $x = 0$ and rises to a further maximum for some $x > 0$ before falling increasingly steeply. Its derivative graph is given in B.





Exercise 9C

31 $d = 6t - 4t^{-1}$

$$\frac{\mathrm{d}d}{\mathrm{d}t} = 6 + 4t^{-2} = 6 + \frac{4}{t^2}$$

32 $q = m + 2m^{-1}$

$$\frac{\mathrm{d}q}{\mathrm{d}m} = 1 - 2m^{-2} = 1 - \frac{2}{m^2}$$

33 $E = \frac{3}{2}kT$

$$\frac{\mathrm{d}E}{\mathrm{d}T} = \frac{3}{2}k$$

34 Let $\mathrm{f}(x) = x^2 - x$

Then $\mathrm{f}'(x) = 2x - 1$

$\mathrm{f}(x)$ is increasing when $\mathrm{f}'(x) > 0$

$2x - 1 > 0$ so $x > \frac{1}{2}$

35 Let $\mathrm{f}(x) = x^2 + bx + c$

Then $\mathrm{f}'(x) = 2x + b$

$\mathrm{f}(x)$ is increasing when $\mathrm{f}'(x) > 0$

$2x + b > 0$ so $x > -\frac{b}{2}$

36 $x + y = 8$ so $y = 8 - x$

Then $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$

37 $x^3 + y = x$ so $y = x - x^3$

Then $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - 3x^2$

38 $V = kr^{-1}$

a Force $= \frac{\mathrm{d}V}{\mathrm{d}r} = -kr^{-2}$

b Rearranging the original formula, $r^{-1} = \frac{V}{k}$. Substituting into the answer from part **a:**

Force $= -k\left(\frac{V}{k}\right)^2 = -\frac{V^2}{k}$

c Let the distance to Alpha be d; then the distance to Omega is $2d$.

Using the formula from part **a**,

$$\frac{\text{Force on Alpha}}{\text{Force on Omega}} = \frac{-kd^{-2}}{-k(2d)^{-2}} = 4$$

39 **a** $A = 2 + qL + qL^2$ so $\frac{\mathrm{d}A}{\mathrm{d}L} = q + 2qL$

b In the context, one would expect that the reading age rating of a book would increase with longer sentences. Since the function is only defined for positive values of L, the function is increasing as long as $q > 0$.

40 $f(x) = 4x^3 + 7x - 2$

So $f'(x) = 12x^2 + 7 > 0$ for all x, since square numbers can never be negative.

Since $f'(x) > 0$ for all x, the function $f(x)$ is increasing for all x.

Exercise 9D

28 a $y = x^4 - x$ so $\frac{dy}{dx} = 4x^3 - 1$

b When $x = 0$, $\frac{dy}{dx} = 4(0)^3 - 1 = -1$

c Then the gradient of the normal at $x = 0$ is 1.

$y(0) = 0$ so the normal passes through the origin and has gradient 1 so has equation $y = x$.

29 a $f(x) = x^3 + x^{-1}$ so $f'(x) = 3x^2 - x^{-2}$

b $f'(1) = 3 - 1 = 2$ so the gradient of the tangent at $x = 1$ is 2.

c $f(1) = 1 + 1 = 2$

The tangent passes through $(1,2)$ and has gradient 2 so has equation $y - 2 = 2(x - 1)$ which rearranges to $y = 2x$, and so the tangent passes through the origin.

30 Let $f(x) = x\sqrt{x+1}$.

From the GDC: $f'(3) = \frac{d}{dx}\left(x\sqrt{x+1}\right)\Big|_{x=3} = \frac{11}{4}$

$f(3) = 6$ so the tangent line passes through $(3,6)$ with gradient $\frac{11}{4}$.

The tangent equation is $y - 6 = \frac{11}{4}(x - 3)$ which rearranges to $y = \frac{11}{4}x - \frac{9}{4}$.

31 Let $f(x) = \frac{1}{x+4}$.

From the GDC: $f'(-2) = \frac{d}{dx}\left(\frac{1}{x+4}\right)\Big|_{x=-2} = -\frac{1}{4}$

$f(-2) = \frac{1}{2}$ so the tangent line passes through $\left(-2, \frac{1}{2}\right)$ with gradient $-\frac{1}{4}$.

Then the normal line passes through $\left(-2, \frac{1}{2}\right)$ with gradient 4.

The normal equation is $y - \frac{1}{2} = 4(x + 2)$ which rearranges to $y = 4x - \frac{17}{2}$.

32 Let $f(x) = x^2e^x$.

From the GDC: $f'(0) = \frac{d}{dx}(x^2e^x)\Big|_{x=0} = 0$

$f(0) = 0$ so the tangent line passes through the origin with gradient 0.

The normal is therefore the y-axis, with equation $x = 0$.

33 $y = x^2$ so $\frac{dy}{dx} = 2x$.

The gradient at $x = 1$ is 2, and the curve passes through $(1,1)$.

The normal has gradient $-\frac{1}{2}$ and passes through $(1,1)$ so has equation $y - 1 = -\frac{1}{2}(x-1)$ which rearranges to $y = \frac{3}{2} - \frac{1}{2}x$.

Substituting back into the original curve equation:

$$x^2 = \frac{3}{2} - \frac{1}{2}x$$
$$2x^2 + x - 3 = 0$$

One intersection is already known: $(1,1)$, so $(x-1)$ is a factor of the quadratic.

$$(x-1)(2x+3) = 0$$

The other intersection is at $\left(-\frac{3}{2}, \frac{9}{4}\right)$.

34 $y = x^{-1}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = -x^{-2}$.

The gradient at $x = 2$ is $-\frac{1}{4}$, and the curve passes through $\left(2, \frac{1}{2}\right)$.

The normal has gradient 4 and passes through $\left(2, \frac{1}{2}\right)$ so has equation $y - \frac{1}{2} = 4(x-2)$ which rearranges to $y = 4x - \frac{15}{2}$.

Substituting back into the original curve equation:

$$\frac{1}{x} = 4x - \frac{15}{2}$$
$$8x^2 - 15 - 2 = 0$$

One intersection is already known: $\left(2, \frac{1}{2}\right)$, so $(x-2)$ is a factor of the quadratic.

$$(x-2)(8x+1) = 0$$

The other intersection is at $\left(-\frac{1}{8}, -8\right)$.

35 $y = x^2$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$.

Require the tangent to have gradient -2 so $2x = -2$ from which $x = -1$.

At $x = -1$, $y = 1$ so the tangent passes through $(-1, 1)$ and has gradient -2.

The equation is $y - 1 = -2(x+1)$ which rearranges to $y = -2x - 1$ or $y + 2x = -1$.

36 $y = x^2 + 2x$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 2$.

Require the normal to have gradient $\frac{1}{4}$ so the tangent has gradient -4 so $2x + 2 = -4$ from which $x = -3$.

At $x = -3$, $y = 3$ so the normal passes through $(-3, 3)$ and has gradient $\frac{1}{4}$.

The equation is $y - 3 = \frac{1}{4}(x+3)$ which rearranges to $y = \frac{1}{4}x + \frac{15}{4}$.

37 $y = x^3$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$.

Require the tangent to have gradient 3 so $3x^2 = 3$ from which $x = \pm 1$.

At $x = -1$, $y = -1$ so the normal passes through $(-1, -1)$ and has gradient $-\frac{1}{3}$.

The equation is $y + 1 = -\frac{1}{3}(x + 1)$ which rearranges to $y = -\frac{1}{3}x - \frac{4}{3}$.

At $x = 1$, $y = 1$ so the normal passes through $(1, 1)$ and has gradient $-\frac{1}{3}$.

The equation is $y - 1 = -\frac{1}{3}(x - 1)$ which rearranges to $y = -\frac{1}{3}x + \frac{4}{3}$.

The two normals have equations $y = -\frac{1}{3}x \pm \frac{4}{3}$.

38 $y = x^3 - x$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 1$.

Require the tangent to have gradient 11 so $3x^2 - 1 = 11$ from which $x = \pm 2$.

At $x = -2$, $y = -6$ so one such tangent passes through $(-2, -6)$.

At $x = 2$, $y = 6$ so the other such tangent passes through $(2, 6)$.

39 $y = x^2 + 4x + 1$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 4$.

Require the tangent to have gradient equal to the y-coordinate.

Substituting: $2x + 4 = x^2 + 4x + 1$

$$x^2 + 2x - 3 = 0$$
$$(x + 3)(x - 1) = 0$$

So $x = -3$ or 1

When $x = -3$, $y = -2$ and when $x = 1$, $y = 6$ so the points are $(-3, -2)$ and $(1,6)$.

40 $y = x^{-1}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = -x^{-2}$.

Let point P on the curve have x-coordinate p so P is $\left(p, \frac{1}{p}\right)$.

Then the tangent at P has gradient $-\frac{1}{p^2}$, so the tangent equation is $y - \frac{1}{p} = -\frac{1}{p^2}(x - p)$.

Given this passes through $(4,0)$, substitute $x = 4, y = 0$ into the tangent equation to find p:

$$-\frac{1}{p} = -\frac{1}{p^2}(4 - p)$$
$$\frac{4}{p^2} = \frac{2}{p}$$
$$p = 2$$

So point P has coordinates $\left(2, \frac{1}{2}\right)$.

Tip: When faced with a question of this sort, where you have a condition you cannot immediately apply (in this case that the tangent line passes through $(4,0)$), it is almost always best to assign an unknown value to the independent variable and calculate everything in terms of that unknown. You will end with an equation into which you can substitute your condition and then solve for the unknown.

41 $y = 4x^{-1}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = -4x^{-2}$.

Let point P on the curve have x-coordinate p so P is $\left(p, \frac{4}{p}\right)$.

Then the tangent at P has gradient $-\frac{4}{p^2}$, so the tangent equation is $y - \frac{4}{p} = -\frac{4}{p^2}(x - p)$.

Given this passes through (1,3), substitute $x = 1, y = 3$ into the tangent equation to find p:

$$3 - \frac{4}{p} = -\frac{4}{p^2}(1 - p)$$
$$\frac{4}{p^2} - \frac{8}{p} + 3 = 0$$
$$3p^2 - 8p + 4 = 0$$
$$(3p - 2)(p - 2) = 0$$

So $p = \frac{2}{3}$ or 2.

42 $y = x^2$ so $\frac{dy}{dx} = 2x$.

Let point P on the curve have x-coordinate p so P is (p, p^2).

Then the tangent at P has gradient $2p$, so the tangent equation is $y - p^2 = 2p(x - p)$.

Given this passes through (2,3), substitute $x = 2, y = 3$ into the tangent equation to find p:

$$3 - p^2 = 2p(2 - p)$$
$$p^2 - 4p + 3 = 0$$
$$(p - 1)(p - 3) = 0$$

$p = 1$ or 3 so the coordinates of P are (1,1) or (3,9).

43 $y = x^{-1}$ so $\frac{dy}{dx} = -x^{-2}$.

Let A be the point with coordinates $\left(a, \frac{1}{a}\right)$.

Then the tangent at A has gradient $-\frac{1}{a^2}$, so the tangent equation is $y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$.

This tangent intersects the x-axis when $y = 0$: $-\frac{1}{a} = \frac{1}{a} - \frac{1}{a^2}x$ so $x = 2a$.

Then Q has coordinates $(2a, 0)$.

The tangent intersects the y-axis when $x = 0$: $y - \frac{1}{a} = \frac{1}{a}$ so $y = \frac{2}{a}$.

Then P has coordinates $\left(0, \frac{2}{a}\right)$.

The area of right-angled triangle OPQ is $\frac{1}{2} \times OP \times OQ = \frac{1}{2} \times \frac{2}{a} \times 2a = 2$, which is independent of a as required.

44 $y = ax^{-1}$ so $\frac{dy}{dx} = -ax^{-2}$.

Point P has coordinates $\left(2, \frac{a}{2}\right)$, so the gradient at P is $-\frac{a}{4}$.

The normal at P therefore has gradient $\frac{4}{a}$ which must equal $\frac{1}{5}$.

The gradient of line $y = \frac{1}{5}x - 2$.

$\frac{4}{a} = \frac{1}{5}$ so $a = 20$.

45 $y = x^3 + x + 1$ so $\frac{dy}{dx} = 3x^2 + 1$.

Let point P have x-coordinate k so P is given by $(k, k^3 + k + 1)$.

Then the tangent at P has gradient $3k^2 + 1$ so the equation of the tangent is

$$y - (k^3 + k + 1) = (3k^2 + 1)(x - k)$$

To find where this tangent intersects the curve, substitute back into the original equation:

$$(3k^2 + 1)(x - k) + k^3 + k + 1 = x^3 + x + 1$$

By construction, this equation must have solution $x = k$ as a repeated root (the tangent touches the curve at $x = k$), so $(x - k)^2$ is a factor of the equation.

$$-2k^3 + 3k^2x + x + 1 = x^3 + x + 1$$
$$x^3 - 3k^2x + 2k^3 = 0$$

By construction, this equation must have solution $x = k$ as a repeated root (the tangent touches the curve at $x = k$), so $(x - k)^2$ is a factor of the equation.

$$(x - k)(x^2 + kx - 2k^2) = 0$$
$$(x - k)(x - k)(x + 2k) = 0$$

So the tangent touches the curve at $x = k$ and intersects it at $x = -2k$.

Mixed Practice

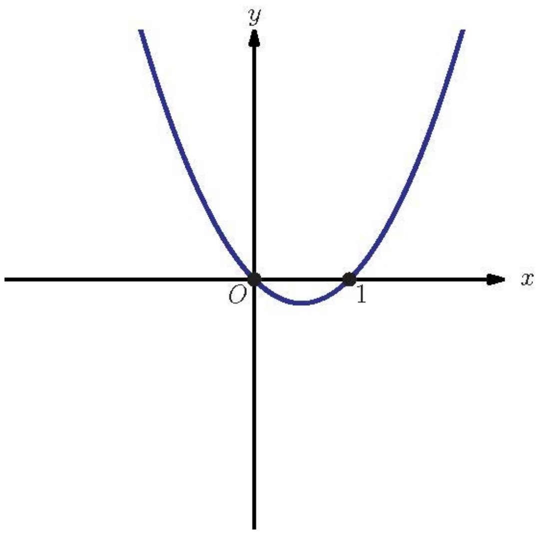
1 **a** $y = 4x^2 - x$

$$\frac{dy}{dx} = 8x - 1$$

b Require $8x - 1 = 15$ so $x = 2$.

When $x = 2, y = 14$ so the point is $(2, 14)$.

2 **a** $f(x) = x^2 - x$



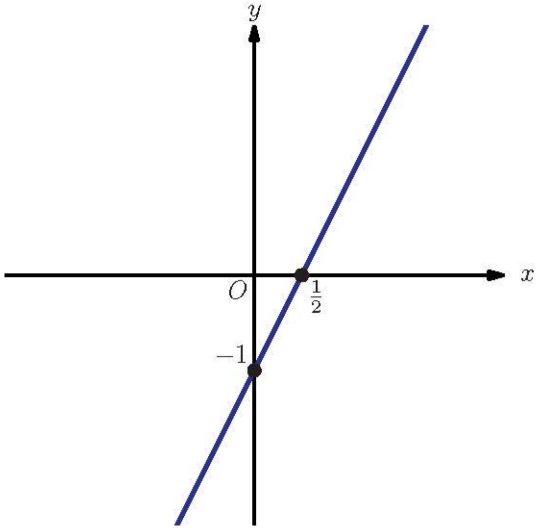
b $f(x) = x^2 - x$ so $f'(x) = 2x - 1$

The curve is increasing when $f'(x) > 0$ so $2x - 1 > 0$

The curve is increasing for $x > \frac{1}{2}$.

Worked solutions

c $f'(x) = 2x - 1$



3 **a** $f(x) = 2x^3 + 5x^2 + 4x + 3$

So $f'(x) = 6x^2 + 10x + 4$

b Then $f'(-1) = 6 - 10 + 4 = 0$

c The tangent is horizontal at $(-1,2)$ so has equation $y = 2$.

4

x	$\frac{\ln(1+x) - x}{x^2}$
1	−0.38605
0.5	−0.37814
0.1	−0.46898
0.01	−0.49669
0.0001	−0.49997

The limit appears to equal −0.5.

5 $y = x^3 - 4$ so $\frac{dy}{dx} = 3x^2$.

When $y = 23$, $x^3 = 27$ so $x = 3$.

At $(3, 23)$, the gradient is $3(3^2) = 27$ so the tangent has equation $y - 23 = 27(x - 3)$.

This rearranges to $y = 27x - 58$.

6 **a** $V = 50 + 12t + 5t^2$ so $\frac{dV}{dt} = 12 + 10t$.

b $V(6) = 302, \frac{dV}{dt}(6) = 72$

After 6 minutes, there is 302 m^3 of water in the tank, and the volume of water in the tank is increasing at a rate of 72 m^3 per minute.

c $\frac{dV}{dt}(10) = 112$, so the volume is increasing faster after 10 minutes than after 6 minutes.

7 **a** $A = 2t - t^2$ so $\frac{dA}{dt} = 2 - 2t$

b **i** $A(5) = 0.75$

ii $\frac{dA}{dt}(0.5) = 1$

c A is increasing when $\frac{dA}{dt} > 0$

$2 - 2t > 0$ for $t < 1$ so the accuracy is increasing for $0 < t < 1$.

Tip: Be careful to check the context and question wording for the domain of the function; here A is only defined for $0 < t < 2$ so it would be incorrect to say that accuracy is increasing for $t < 1$.

8 $f(x) = \frac{1}{2}x^2 - 8x^{-1}, x \neq 0$

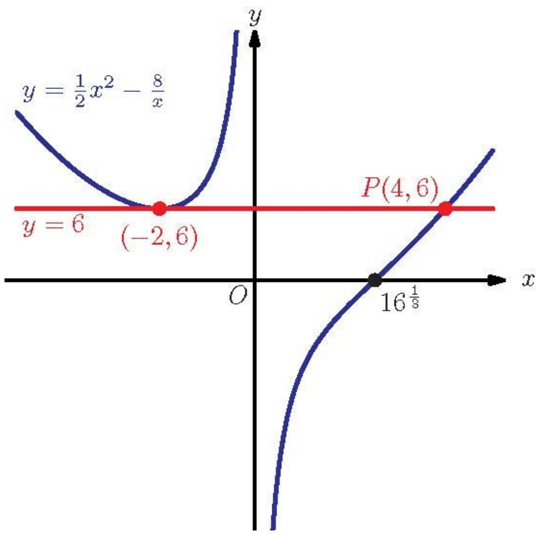
a $f(-2) = 6$

b $f'(x) = x + 8x^{-2}$

c $f'(-2) = -2 + 2 = 0$

d T is a line passing through $(-2,6)$ with gradient 0 so has equation $y = 6$.

e, f



g From GDC: the tangent intersects the graph again at $P(4, 6)$.

9 $y = 2x^3 - 8x + 3$ so $\frac{dy}{dx} = 6x^2 - 8$.

Require that the gradient is -2 so $6x^2 - 8 = -2$.

$6x^2 = 6$ so $x = \pm 1$.

When $x = -1$, $y = 9$ and when $x = 1$, $y = -3$ so the two points are $(-1,9)$ and $(1,-3)$.

10 **a** $y = x^3 - 6x^2$ so $\frac{dy}{dx} = 3x^2 - 12x$.

Gradient is zero where $3x^2 - 12x = 0$.

$3x(x - 4) = 0$ so $x = 0$ or $x = 4$.

When $x = 0$, $y = 0$ and when $x = 4$, $y = -32$ so the points are $(0,0)$ and $(4,-32)$.

b The line passing through the origin and $(4, -32)$ is $y = -8x$.

11 a $y = 3x^2 + 6x$ so $\frac{dy}{dx} = 6x + 6$.

$y = 3x(x + 2)$ has roots $x = 0$ and $x = -2$.

When $x = 0$, $y = 0$ and $\frac{dy}{dx} = 6$ so the tangent has equation $y = 6x$.

When $x = -2$, $y = 0$ and $\frac{dy}{dx} = -6$ so the tangent has the equation:

$y = -6(x + 2) = -6x - 12$

b Substituting to find the intersection:

$6x = -6x - 12$ so $12x = -12$, and then $x = -1$. The two lines intersect at $(-1, -6)$.

12 $y = 2x^2 + c$ so $\frac{dy}{dx} = 4x$.

If the gradient at $(p, 5)$ is -8 then $4p = -8$ so $p = -2$.

Substituting $(-2,5)$ into the curve equation: $5 = 8 + c$ so $c = -3$.

13 a $P(t) = 15t^2 - t^3$ so $\frac{dP}{dt} = 30t - 3t^2$.

b $\frac{dP}{dt}(6) = 72$ and $\frac{dP}{dt}(12) = -72$

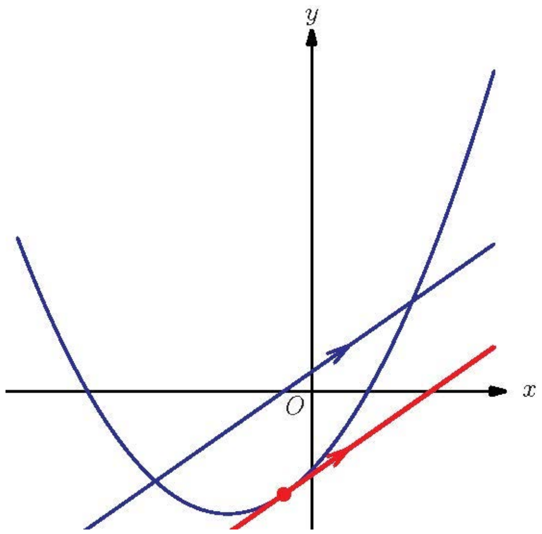
c The profit is increasing by \$72 per month after 6 months, but is falling at a rate of \$72 per month after 12 months.

14 a i $f(x) = 2x + 1$ so $f'(x) = 2$.

ii $g(x) = x^2 + 3x - 4$ so $g'(x) = 2x + 3$.

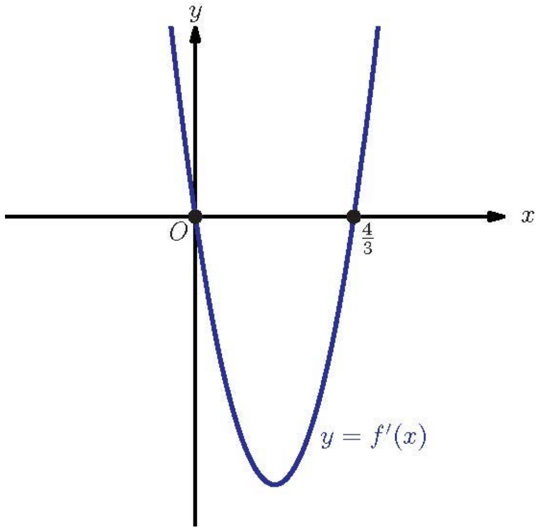
b If the two graphs have the same gradient for a value of x then $2x + 3 = 2$ so $x = -\frac{1}{2}$.

c The tangent line is parallel to the line $y = f(x)$.



15 a i The graph is decreasing for $0 < x < \frac{4}{3}$

ii $f(x)$ has stationary points at $x = 0$ and $x = \frac{4}{3}$, so these are the roots fo $f'(x)$. The graph is increasing for $x < 0$ and $x > \frac{4}{3}$, so $f'(x)$ is positive in these regions, and negative in $0 < x < \frac{4}{3}$.

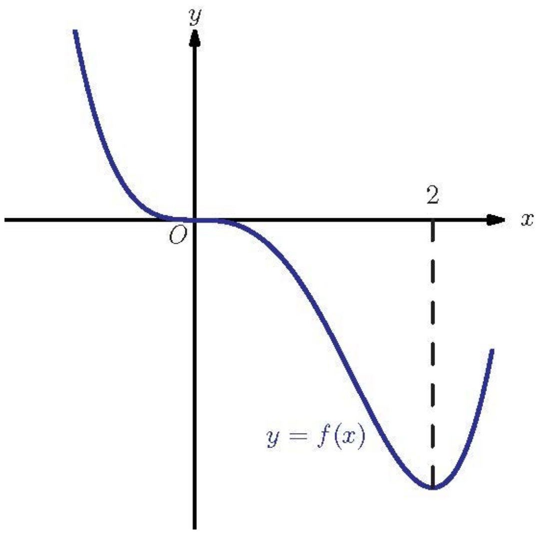


b i $f(x)$ is decreasing when $f'(x) < 0$ which is for $x < 0$ and $0 < x < 2$.

ii Stationary points occur when $f'(x) = 0$: at $x = 0$ and $x = 2$.

$x = 0$ will be a horizontal inflexion point, with negative gradient either side.

$x = 2$ will be a minimum point, with the gradient changing from negative to positive.



16 a Line connecting $(-1, -2)$ and $(1, 4)$: Gradient $= \frac{4-(-2)}{1-(-1)} = 3$.

b $f(x) = x^2 - x + 2$ so $f'(x) = 2x - 1$.

c Require that $2x - 1 = 3$ so $x = 2$.

$f(2) = 4$ so at point $(2,4)$, the curve $y = f(x)$ is parallel to line l.

d Require that $2x - 1 = -\frac{1}{3}$ so $x = \frac{1}{3}$.

$f\left(\frac{1}{3}\right) = \frac{16}{9}$ so at point $\left(\frac{1}{3}, \frac{16}{9}\right)$, the curve $y = f(x)$ is perpendicular to line l.

e $f(3) = 5$ so the gradient at $(3,8)$ is 5.

The tangent has equation $y - 8 = 5(x - 3)$ which rearranges to $y = 5x - 7$.

f The vertex occurs when $f'(x) = 0$.

$2x - 1 = 0$ so $x = \frac{1}{2}$.

$f\left(\frac{1}{2}\right) = \frac{7}{4}$ so the vertex is at $\left(\frac{1}{2}, \frac{7}{4}\right)$, and the gradient is zero at this point.

17 a $f(x) = x^2 + x - 5$ so $f'(x) = 2x + 1$.

b Substituting: $2x + 1 = x^2 + x - 5$

$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$

$x = 3$ or $x = -2$

18 $y = ax^2 + bx$ so $\frac{dy}{dx} = 2ax + b$.

Substituting $x = 2, y = -2$ into the curve equation:

$-2 = 4a + 2b$ (1)

Substituting $x = 2, \frac{dy}{dx} = 3$ into the gradient equation:

$3 = 4a + b$ (2)

(1) − (2): $-5 = b$

So $b = -5, a = 2$

19 $y = ax^2 + bx$ so $\frac{dy}{dx} = 2ax + b$.

Substituting $x = 1, y = 5$ into the curve equation:

$5 = a + b$ (1)

If the normal gradient is $\frac{1}{3}$ then the curve gradient is -3.

Substituting $x = 1, \frac{dy}{dx} = -3$ into the gradient equation:

$-3 = 2a + b$ (2)

(2) − (1): $-8 = a$

So $a = -8, b = 13$

20 $y = 5x^2 - 4$ so $\frac{dy}{dx} = 10x$.

When $x = 1$, $y = 1$ and the gradient is 10.

The equation of the tangent at $(1, 1)$ is $y - 1 = 10(x - 1)$ which rearranges to $y = 10x - 9$

When $x = 2$, $y = 16$ and the gradient is 20.

The equation of the tangent at $(2,16)$ is $y - 16 = 20(x - 2)$ which rearranges to $y = 20x - 24$

At the intersection of these lines, $10x - 9 = 20x - 24$

$10x = 15$ so $x = \frac{3}{2}, y = 6$. The two tangents intersect at $\left(\frac{3}{2}, 6\right)$

21 $f(x) = \frac{\ln(4x)}{x}$

From the GDC: $f'(0.25) = 16$ so the tangent at $P(0.25,0)$ is $y = 16\left(x - \frac{1}{4}\right) = 16x - 4$

This is perpendicular to the tangent at Q so the tangent at Q has gradient $-\frac{1}{16}$.

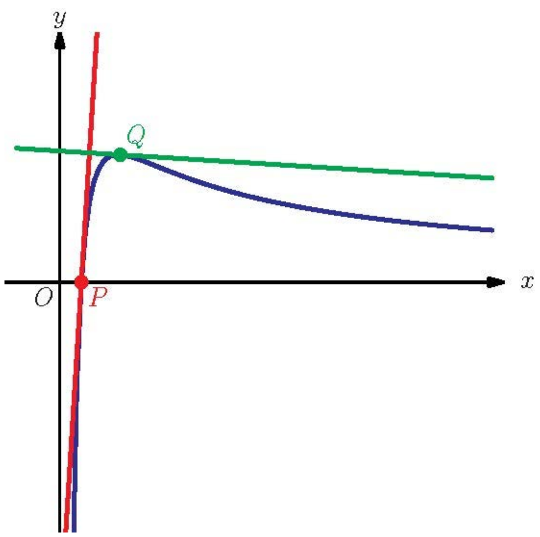
The graph of $f(x)$ increases to a maximum at $x = 0.68$ and then decreases, with gradient decreasing towards zero.

From the GDC: $f'(5) < -\frac{1}{16}$ so the only point within the domain where the tangent is perpendicular to the tangent at $x = \frac{1}{4}$ must be close to the stationary point.

x	$-\frac{1}{f'(x)}$
0.8	3.92
0.75	5.70
0.7	16.54
0.71	11.50
0.705	13.53
0.703	14.58
0.702	15.17
0.701	15.83
0.7005	16.17

The x-coordinate of Q must lie between 0.7 and 0.7005, so to 3DP, the x-coordinate is 0.7.

Q has coordinates $(0.7,1.47)$.



10 Core: Integration

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 10A

23

$$\int \frac{4}{3t^2} - \frac{2}{t^5}\,\mathrm{d}t = \int \frac{4}{3}t^{-2} - 2t^{-5}\,\mathrm{d}t$$
$$= -\frac{4}{3}t^{-1} + \frac{2}{4}t^{-4} + c$$
$$= -\frac{4}{3}t^{-1} + \frac{1}{2}t^{-4} + c$$

24

$$y = \int 3x^2 - 4\ \mathrm{d}x$$
$$= \frac{3x^3}{3} - \frac{4x}{1} + c$$
$$= x^3 - 4x + c$$

When $x = 1, y = 4$ so,

$$4 = 1 - 4 + c$$

So, $c = 7$ and then $y = x^3 - 4x + 7$

25

$$y = \int 4x^{-2} - 3x^2\ \mathrm{d}x$$
$$= \frac{4x^{-1}}{-1} - \frac{3x^3}{3} + c$$
$$= -\frac{4}{x} - x^3 + c$$

When $x = 2, y = 0$ so,

$$0 = -\frac{4}{2} - 2^3 + c$$

So, $c = 10$ and then $y = -\frac{4}{x} - x^3 + 10$

26

$$\int (3x-2)(x^2+1)\,\mathrm{d}x = \int 3x^3 - 2x^2 + 3x - 2\ \mathrm{d}x$$
$$= \frac{3x^4}{4} - \frac{2x^3}{3} + \frac{3x^2}{2} - 2x + c$$

27

$$\int z^2\left(z+\frac{1}{z}\right)\mathrm{d}z = \int z^3 + z\,\mathrm{d}z$$
$$= \frac{z^4}{4} + \frac{z^2}{2} + c$$

28

$$\int \frac{x^5 - 2x}{3x^3}\mathrm{d}x = \int \frac{x^2}{3} - \frac{2x^{-2}}{3}\,\mathrm{d}x$$
$$= \frac{x^3}{9} + \frac{2x^{-1}}{3} + c$$
$$= \frac{x^3}{9} + \frac{2}{3x} + c$$

29 a When $t = 2, \frac{\mathrm{d}m}{\mathrm{d}t} = 0.5$ so $0.5 = 2k + 0.1$

$k = 0.2$

b Take $m(0) = 0$ since initially the mass is negligible.

$$m = \int kt + 0.1\ \mathrm{d}t$$
$$= \frac{1}{2}kt^2 + 0.1t + c$$

Substituting $k = 0.2$ and $m(0) = 0$ so $c = 0$:

$$m(t) = 0.1t^2 + 0.1t$$

Then $m(5) = 0.1 \times 25 + 0.1 \times 5 = 3$ kg

30 a Let the volume of water in the bath be given by V.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{80}{t^2}$$

$V(1) = 0$:

$$V = \int 80t^{-2}\ \mathrm{d}t$$
$$= -80t^{-1} + c$$

$V(1) = 0$:

$$0 = -80 + c \Rightarrow c = 80$$
$$V = 80\left(1 - \frac{1}{t}\right)$$

When $t = 2, V = 40$ l

b When $V = 60, 60 = 80\left(1 - \frac{1}{t}\right)$

$$\frac{1}{t} = \frac{1}{4}$$

$t = 4$; after 4 minutes, the bath holds 60 l.

c A graph of the curve shows that it has an asymptote at $V = 80$ and it will never exceed this value, so the bath will never overflow.

Exercise 10B

5 From GDC:

$$\int_2^5 2x + \frac{1}{x^2}\,\mathrm{d}x = 21.3 \text{ (to 3 s.f.)}$$

6 From GDC:

$$\int_1^2 (x-1)^3\,\mathrm{d}x = 0.25$$

7 From GDC:

$$\int_1^4 x^2 + 3\,\mathrm{d}x = 30$$

8 From GDC:

$$\int_{0.5}^2 4 - x^{-2}\,\mathrm{d}x = 4.5$$

9 From GDC plot, intercepts between x-axis and the graph are at $(2,0)$ and $(6,0)$

Then from GDC, the area is

$$\left|\int_2^6 -x^2 + 8x - 12\,\mathrm{d}x\right| = \frac{32}{3}$$

10 From GDC:

$$\int_0^3 9 - x^2\,\mathrm{d}x = 18$$

11 a From GDC, the two graphs intersect at the origin and at $(2.5, 6.25)$

b Shaded area $= \int_0^{2.5}(5x - x^2) - \frac{5x}{2}\,\mathrm{d}x = 2.60$

12 When $t = 2$, $5 + kt^2 = 9$

$5 + 4k = 9$ so $k = 1$

Total filtered in the first two minutes is

$$\int_0^2 5 + t^2\,\mathrm{d}t = \frac{38}{3}\text{ l}$$

13 Let $p(t)$ be the amount of paint sprayed in grams per second at time t seconds.

$p = kt$ where k is the constant of proportionality in the system.

$p(10) = 20$ so $k = 2$

$$\int_0^{60} p\,\mathrm{d}t = \int_0^{60} 2t\,\mathrm{d}t$$
$$= 3600\text{ g}$$

14 a Let $v(t)$ be the rate of sand falling at time t seconds.

$$v = 100t^{-3}$$

Amount of sand after 5 seconds is given by

$$\int_0^5 v \, \mathrm{d}t = 10 + \int_1^5 100t^{-3} \, \mathrm{d}t$$
$$= 58 \text{ g}$$

b From the calculator, the amount which will eventually fall is

$$10 + \int_1^{\infty} 100t^{-3} \, \mathrm{d}t = 60 \text{ g}$$

c The model suggests that it takes infinitely long for all 60 g to fall through the timer, and yet is always falling; this in turn suggests that sand is infinitely divisible.

The model also suggests that at the start, the rate at which the sand falls is infinite.

15 $f'(x) = x^2$ so

$$f(x) = \int x^2 \, \mathrm{d}x$$
$$= \frac{1}{3}x^3 + c$$

Given $f(0) = 4$, $c = 4$.

$$\int_0^3 f(x) \, \mathrm{d}x = \int_0^3 \frac{1}{3}x^3 + 4 \, \mathrm{d}x$$
$$= 18.75$$

16 $\int f(x) \, \mathrm{d}x = 4(x^3 + x^{-2} + c)$

Then

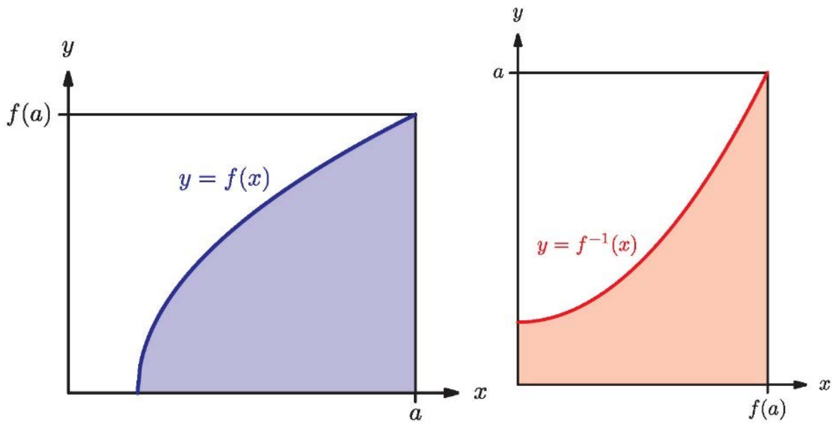
$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}(4(x^3 + x^{-2} + c)$$
$$= 12x^2 - 8x^{-3}$$

17 This is the same as the area enclosed by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$, by symmetry.

$$\int_0^4 \sqrt{x} \, \mathrm{d}x = \int_0^4 x^{\frac{1}{2}} \, \mathrm{d}x$$
$$= \frac{16}{3}$$

18 Since the graph of $f^{-1}(x)$ is the reflection of $f(x)$ in the line $y = x$,

$$\int_0^a \mathrm{f}(x) \, \mathrm{d}x + \int_0^{f(a)} f^{-1}(x) \, \mathrm{d}x = af(a)$$



where $af(a)$ is the area of the rectangle bordering the limits of the integral area, which would have vertices at the origin, $(0, f(a))$, $(a, 0)$ and $(a, f(a))$.

Then

$$\int_0^{f(a)} f^{-1}(x)\ \mathrm{d}x = af(a) - A$$

Mixed Practice 10

1 $\int x^3 - 3x^{-2}\ \mathrm{d}x = \frac{x^4}{4} + 3x^{-1} + c$

2 $\int 4x^2 - 3x + 5\ \mathrm{d}x = \frac{4}{3}x^3 - \frac{3}{2}x^2 + 5x + c$

3 $\int_1^5 2x^{-4}\ \mathrm{d}x = 0.661$

4 $y = \int 3x^2 - 8x\ \mathrm{d}x = x^3 - 4x^2 + c$

$y(1) = 3 = 1 - 4(1) + c$

$c = 6$

$y = x^3 - 4x^2 + 6$

5 $I = \int_2^a 2 - \frac{8}{x^2}\mathrm{d}x = 9$

From GDC:

When $a = 5, I = 3.6$

When $a = 7, I = 7.14$

When $a = 8, I = 9$

$a = 8$

Tip: If you learn further integration methods, you will find how to calculate this algebraically.

6 $I = \int_1^b 9x - x^2 - 8\,\mathrm{d}x = 42.7$

From GDC:

When $b = 5, I = 34.7$

When $b = 6, I = 45.8$

When $b = 5.5, I = 40.5$

When $b = 5.7, I = 42.71$

When $b = 5.65, I = 42.2$

So the value of b for which $I = 42.7$ is between 5.65 and 5.7, so to 1 decimal place, $b = 5.7$

7 **a** From calculator, solutions to $-x^3 + 9x^2 - 24x + 20 = 0$ are $x = 2, 5$

So P has coordinates $(2, 0)$ and Q has coordinates $(5, 0)$

b From calculator,

$$\int_2^2 -x^3 + 9x^2 - 24x + 20\,\mathrm{d}x = 6.75$$

8 **a** $y = 0.2x^2$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.4x$

On the curve, at $x = 4$, $y = 0.2 \times 4^2 = 3.2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1.6$

So the tangent has gradient 1.6 and passes through $(4, 3.2)$

Tangent equation is $y - 3.2 = 1.6(x - 4)$

$y = 1.6x - 3.2$

b Substituting $x = 2$ into the tangent equation gives $y = 3.2 - 3.2 = 0$ so the x-intercept is at $(2, 0)$.

c

$$\text{Shaded area} = \int_0^4 0.2x^2\,\mathrm{d}x - \int_2^4 1.6x - 3.2\,\mathrm{d}x$$
$$= 1.07$$

9 **a** $y = x^{-2}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = -2x^{-3}$

When $x = 1$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -2$

Then the normal at $(1, 1)$ has gradient $\frac{1}{2}$.

Normal has equation $y - 1 = \frac{1}{2}(x - 1)$

$$y = \frac{1}{2}x + \frac{1}{2}$$

When $x = -1$, $y = -\frac{1}{2} + \frac{1}{2} = 0$ so the x-axis intercept of the normal is at $(-1, 0)$.

b

$$\text{Shaded area} = \int_{-1}^1 \frac{1}{2}x + \frac{1}{2}\,\mathrm{d}x - \int_1^2 x^{-2}\,\mathrm{d}x$$
$$= 1.5$$

10 If the gradient of the normal is always x^2 then $f'(x) = -\frac{1}{x^2} = -x^{-2}$

$$\begin{aligned} f(x) &= \int -x^{-2}\ \mathrm{d}x \\ &= x^{-1} + c \end{aligned}$$

$f(1) = 2 = 1 + c$ so $c = 1$

$$y = 1 + \frac{1}{x}$$

Then $f(2) = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$

11 Let $V(t)$ be the volume of water in the container at time t, measured in cm^3.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 20t$$

$$\begin{aligned} V &= \int 20t\ \mathrm{d}t \\ &= 10t^2 + c \end{aligned}$$

If $V(0) = 0$ then $0 = 0 + c$ so $c = 0$

$$V(t) = 10t^2$$

Then $V(10) = 1000\ \mathrm{cm}^3$

12 Let $M(t)$ be the mass of the puppy at A months, measured in kg.

$$M(6) = 2.3$$

Then $\frac{dM}{dA} = \frac{A}{20} + c$

When $A = 10$, $\frac{\mathrm{d}M}{\mathrm{d}A} = 1.5 = \frac{1}{2} + c$ so $c = 1$

$$\frac{dM}{dA} = \frac{A}{20} + 1$$

Then for $A \geqslant 6$,

$$\begin{aligned} M(A) &= M(6) + \int_6^{18} \frac{A}{20} + 1\ \mathrm{d}A \\ &= 21.5\ \mathrm{kg} \end{aligned}$$

13 $f'(x) = 3x^2 + k$

$$\begin{aligned} f(x) &= \int 3x^2 + k\ \mathrm{d}x \\ &= x^3 + kx + c \end{aligned}$$

Substituting:

$f(1) = 13 = 1 + k + c$ (1)

$f(2) = 24 = 8 + 2k + c$ (2)

$(2) - (1): k + 7 = 11$

So $k = 4$ and therefore $c = 8$

Then $f(3) = 3^3 + 4 \times 3 + 8 = 47$

14 a The measurement is of temperature increase in the water, but this is assumed equal to the energy output of the nut – that is, there is an assumption that no energy is lost from the system into the surroundings.

b Let $E(t)$ be the amount of energy (in calories) absorbed by the water by time t seconds.

$$\frac{dE}{dt} = \frac{k}{t^2} \text{ for } t > 1$$

$E(1) = 10$ and $E(2) = 85$

$$\begin{aligned} E(2) &= E(1) + \int_1^2 kt^{-2}\,dt \\ &= 10 + 0.5k \\ &= 85 \end{aligned}$$

Then $0.5k = 75$ so $k = 150$

Total energy after an indefinite burn is

$$\begin{aligned} E(1) + \int_1^\infty kt^{-2}\,dt &= 10 + k \\ &= 160 \text{ calories} \end{aligned}$$

15 $\frac{dy}{dx} = kx^2$

$$\begin{aligned} y &= \int kx^2\,dx \\ &= \frac{k}{3}x^3 + c \end{aligned}$$

Substituting $x = 0, y = 3$ and $x = 1, y = \frac{14}{3}$:

$3 = 0k + c$ so $c = 3$

$\frac{14}{3} = \frac{1}{3}k + 3$ so $k = 5$

Then $y = \frac{5}{3}x^3 + 3$

16 $f'(x) = 3x - x^2$

$$\begin{aligned} f(x) &= \int 3x - x^2\,dx \\ &= \frac{3}{2}x^2 - \frac{1}{3}x^3 + c \end{aligned}$$

Substituting $x = 4, y = 0$:

$0 = \frac{3}{2}(16) - \frac{1}{3}(64) + c$ so $c = \frac{64}{3} - 24 = -\frac{8}{3}$

$$\begin{aligned} \int_0^4 f(x)\,dx &= \int_0^4 \frac{3}{2}x^2 - \frac{1}{3}x^3 - \frac{8}{3}\,dx \\ &= 0 \end{aligned}$$

17

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\ \mathrm{d}x\right)$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}(3x^2 - 2x^{-1} + c)$$
$$= 6x + 2x^{-2}$$

18 a $y = x^2$ so when $y = 9$ for $x > 0$, $x = 3$.

B has coordinates $(3, 0)$.

b Shaded area is the rectangle area less the area under the curve.

$$\text{Shaded area} = 27 - \int_0^3 x^2\ \mathrm{d}x$$
$$= 18$$

11 Applications and interpretation: Number and finance

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 11A

27 a $V = 2.4 \times 5.1 \times 3.8 = 46.512\ \text{cm}^3$

b i $46.51\ \text{cm}^3$

ii $47\ \text{cm}^3$

28 a $r = \ln\left(\frac{8.72^3}{5.3}\right) - \sqrt{5.3} = 2.526978 \ldots \approx 2.5\ (\text{to 2 s. f.})$

b $r \approx \ln\left(\frac{9^3}{5}\right) - \sqrt{5} = 2.746 \ldots \approx 3\ (\text{to 1 s. f.})$

29 a i 128.4

ii 130

b

$$\begin{aligned}\text{Percentage error} &= \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\% \\ &= \frac{|130 - 128.37|}{128.37} \times 100\% \\ &= 1.27\%\end{aligned}$$

30 a $\bar{x} = 329.54\ \text{ml}$

b

$$\begin{aligned}\text{Percentage error} &= \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\% \\ &= \frac{|330 - 329.54|}{329.54} \times 100\% \\ &= 0.140\%\end{aligned}$$

31 $84.5\ \text{cm} \leqslant l < 85.5\ \text{cm}$

32 Lower bound for actual mass is 129.5 g

Minimum mass for six cans is $129.5 \times 6 = 777$ g

33 a $AC^2 = BC^2 - AB^2$

$4.65 \leqslant AB < 4.75$ and $12.25 \leqslant BC < 12.35$

Maximum length of AC occurs when BC is at maximum length and AB is minimum

$\max AC = \sqrt{12.35^2 - 4.65^2} = 11.441$ cm

Minimum length of AC occurs when BC is at minimum length and AB is maximum

$\min AC = \sqrt{12.25^2 - 4.75^2} = 11.292$ cm

Neither of these bounds is achievable, since each is calculated using an unachievable bound of one of the other lengths.

$11.292 \text{ cm} < AC < 11.441 \text{ cm}$

b Assume one bound is 'accurate' and the other an approximation.

$$\text{Percentage error} = \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$

The percentage error will be the same at numerator no matter which way around the values are given, but will be greater overall with the lesser value as denominator.

$$\text{Maximum percentage error} = \frac{|11.441 - 11.292|}{11.292} \times 100\%$$
$$= 1.325\%$$

34 $I = \frac{V}{R}$

$235 \leqslant V < 245, 17.5 \leqslant R < 18.5$

$$\frac{235}{18.5} < I < \frac{245}{17.5}$$

$12.7027 < I < 14$

35 $7.5 \leqslant b < 8.5, 4.5 \leqslant c < 5.5, 1.5 \leqslant d < 2.5$

$$\frac{7.5}{5.5 - 1.5} < a < \frac{8.5}{4.5 - 2.5}$$

$1.875 < a < 4.25$

36 a $7 + 1 \leqslant p + q \leqslant 9 + 5$; central value is $8 + 3 = 11$

$$\text{Percentage error} = \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$

The percentage error will be the same at numerator no matter whether max or min value is used, but will be greater overall with the min value as denominator.

$$\text{Maximum percentage error} = \frac{|11 - 8|}{8} \times 100\%$$
$$= 37.5\%$$

b $7 - 5 \leqslant p + q \leqslant 9 - 1$; central value is $8 - 3 = 5$

$$\text{Percentage error} = \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$

The percentage error will be the same at numerator no matter whether max or min value is used, but will be greater overall with the min value as denominator.

$$\text{Maximum percentage error} = \frac{|5 - 2|}{2} \times 100\%$$
$$= 150\%$$

37 **a** **i** $P(1) = 0.9$

ii $P(2) = 1.2$

iii $P(4.5) = -0.675$

b Any value outside the interval $0 \leqslant P \leqslant 1$ must be incorrect, given the context, so (ii) and (iii) are clearly wrong.

38 **a** $1.445 \leqslant a < 1.455$

b $27.86 \leqslant 10^a < 28.51$

c $10^a = 30$ to 1 s.f. (or to the nearest ten)

It cannot accurately be stated as 28 to 2 s. f., since it is possible to take a value greater than 28.5.

39 Maximum mass of a pallet is 1.8 tonnes

If all pallets are maximal, a truck can carry $\left\lfloor \frac{12}{1.8} \right\rfloor = 6$ pallets

Therefore it is possible that $\left\lceil \frac{52}{6} \right\rceil = 9$ trucks would be needed.

Tip: The rounding brackets $\lfloor \quad \rfloor$ (floor) and $\lceil \quad \rceil$ (ceiling) which refer to the nearest integer when rounding down / up respectively can be very useful for expressing working like this concisely.

40 $9.5 \leqslant w < 15, \quad 25 \leqslant l < 35$

$$9.5 \times 25 \leqslant \text{area} < 15 \times 35$$
$$237.5 \text{ cm}^2 \leqslant \text{area} < 525 \text{ cm}^2$$

Exercise 11B

7 Using GDC:

$n = 20$

$I\% = 4\%$

$PV = 60\,000$

$PMT = ?$

$FV = 0$

$P/Y = 1$

$C/Y = 1$

Solving for PMT: $PMT = 4414.91$

She can withdraw $4414.91 each year.

8 **a** Using GDC:

$n = ?$

$I\% = 3\%$

$PV = 150\,000$

$PMT = -15\,000$

$FV = 0$

$P/Y = 1$

$C/Y = 1$

Solving for n: $n = 12.1$

He can withdraw $15 000 at the end of each year for 12 years.

b Using GDC:

$n = 13$

$I\% = 3\%$

$PV = 150\ 000$

$PMT = -15\ 000$

$FV = ?$

$P/Y = 1$

$C/Y = 1$

Solving for FV: $FV = 13\ 986.80$

That is, in the thirteenth year, he would be £13 986.80 short of the £15 000 he would want; he could only draw out $\$1013.20$.

9 Using GDC:

$n = 10$

$I\% = 2.5\%$

$PV = 0$

$PMT = -1000$

$FV = ?$

$P/Y = 1$

$C/Y = 1$

Solving for FV: $FV = 11\ 203.38$

In total she would have deposited $10 000 so the total interest earned is $1203.38

10 a Using GDC:

Option A: After the first year, the loan amount is $€1500 \times \left(1 + \frac{1}{120}\right)^{12} = €1657.07$

$n = 48$

$I\% = 10\%$

$PV = 1650$

$PMT = ?$

$FV = 0$

$P/Y = 12$

$C/Y = 12$

Solving for PMT: $PMT = €42.03$

Option B:

$n = 60$

$I\% = 10\%$

$PV = 1500$

$PMT = ?$

$FV = 0$

$P/Y = 12$

$C/Y = 12$

Solving for PMT: $PMT = €31.87$

b The total amount paid under plan A is $48 \times €42.03 = \$2017.44$

The total amount paid under plan B is $60 \times €31.87 = \$1912.20$

The difference is $€2017.44 - €1912.20 = €105.24$

c Which plan is better would depend on Anwar's circumstances and the consequent value to him of the payment holiday.

We accept that by taking out a loan – that is, paying interest for the use of a block of capital – can be justified. The payment holiday is simply a different form of loan, in which he will eventually pay €105.24 additional interest for the use of the money for a year without making payments.

11 a For the first two years:

$n = 24$

$I\% = 5\%$

$PV = 10000$

$PMT = -400$

$FV = ?$

$P/Y = 12$

$C/Y = 12$

Solving for FV: $FV = \$975.05$

At the end of the two years, the outstanding loan is $975.05

For the subsequent period:

$n = ?$

$I\% = 8\%$

$PV = 975.05$

$PMT = -400$

$FV = 0$

$P/Y = 12$

$C/Y = 12$

Worked solutions

Solving for n: $n = 2.5$

She will need an additional 3 months to pay of the remainder of the loan.

In total, it will take 27 months to pay the loan.

b

$n = 3$

$I\% = 8\%$

$PV = 975.05$

$PMT = -400$

$FV = ?$

$P/Y = 12$

$C/Y = 12$

Solving for FV: $FV = \$213.35$

So at the end of the third month, when she comes to pay off the balance, \$400 would overpay by \$213.35. That is, the final payment will be \$186.65

12 a **Plan A**:

$n = 12 \times 25 = 300$

$I\% = 3\%$

$PV = 125000$

$PMT = ?$

$FV = 0$

$P/Y = 12$

$C/Y = 12$

Solving for PMT: $PMT = -€592.76$

Plan B:

$n = 12 \times 30 = 360$

$I\% = 3.5\%$

$PV = 125000$

$PMT = ?$

$FV = 0$

$P/Y = 12$

$C/Y = 1$

Solving for PMT: $PMT = -€557.48$

Monthly payments under plan A: €592.76

Monthly payments under plan B: €557.48

Monthly payment is less under plan B.

b Total amount paid under plan A: $592.76 \times 300 = €177\,828$

Total amount paid under plan B: $557.48 \times 360 = €200\,693$

Total repaid is less under plan A.

13 For the first two years:

$n = 24$

$I\% = 5\%$

$PV = 15000$

$PMT = -300$

$FV = ?$

$P/Y = 12$

$C/Y = 12$

Solving for FV: $FV = -9018.34$

After two years, the outstanding balance will be \$9018.34.

For the following three years:

$n = 36$

$I\% = 6.5\%$

$PV = 9018.34$

$PMT = ?$

$FV = 0$

$P/Y = 12$

$C/Y = 12$

Solving for PMT: $PMT = -270.29$

So, paying only \$270.29 per month for the remaining 3 years, Brian would pay off the loan.

Brian can afford this loan.

Mixed Practice

1 **a** $x = 12.5$ (to 1 d. p.)

b $x = 12$ (to 2 s. f.)

2 For side length x: $0.055 \leqslant x < 0.065$

Then $0.055^2 x \leqslant x < 0.065^2$

$$\text{Percentage error} = \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$

Maximum percentage error will occur when the true value is minimum

$$\text{Maximum percentage error} = \frac{|0.06^2 - 0.055^2|}{0.055^2} \times 100\%$$
$$= 19.0\%$$

3

$$\text{Percentage error} = \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$
$$= \frac{|1.4 - \sqrt{2}|}{\sqrt{2}} \times 100\%$$
$$= 1.01\%$$

4

$n = 25$

$I\% = 3\%$

$PV = 60000$

$PMT = ?$

$FV = 10000$

$P/Y = 1$

$C/Y = 1$

Solving for PMT: $PMT = -3719.95$

He can draw £3719.95 per year.

5 **a** 1380 m

b

$$h = 1380 \tan 28.3°$$
$$= 743 \text{ m}$$

c

$$\text{Percentage error} = \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$
$$= \frac{|743 - 718|}{718} \times 100\%$$
$$= 3.49\%$$

6 **a** $p = 1.775 - \frac{\sqrt{1.44}}{48} = 1.75$

b **i** To 1s.f., $x = 2, y = 1, z = 50$

ii $p = 2 - \frac{\sqrt{1}}{50} = 1.98$

c

$$\text{Percentage error} = \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$
$$= \frac{|1.98 - 1.75|}{1.75} \times 100\%$$
$$= 13.1\%$$

7 **a**

$$T = \frac{(\tan(60°) + 1)(2\cos(30°) - 1)}{41^2 - 9^2}$$
$$= \frac{(\sqrt{3}+1)(\sqrt{3}-1)}{(41-9)(41+9)}$$
$$= \frac{2}{32 \times 50}$$
$$= \frac{1}{800} = 0.001\ 25$$

b **i** $T = 0.0013$ (to 2 s. f.)

ii $T = 0.001$ (to 3 d. p.)

c

$$\text{Percentage error} = \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$
$$= \frac{|0.002 - 0.00125|}{0.00125} \times 100\%$$
$$= 60\%$$

8 Age has been given as a birthday rather than an age.

Height is unreasonable (5.11 metres is not human) so likely represents a unit error, and should indicate 5 foot 11 inches. However, since it is possible that it is a different error (for example a digit swap, and should read 1.51 m), the data cannot just be amended and should be discarded.

9 **a** $15.05 \leq C < 15.15,\ \ 4.75 \leq d < 4.85$

$$\pi = \frac{C}{d}$$

$$\frac{15.05}{4.85} < \pi < \frac{15.15}{4.75}$$

$$3.1031 < \pi < 3.1895$$

b Using $C = 15.1, d = 4.8$ she gets estimate $\pi = 3.14583$

$$\text{Percentage error} = \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$
$$= \frac{|3.14583 - \pi|}{\pi} \times 100\%$$
$$= 0.135\%$$

c Using lower bound:

$$\text{Percentage error} = \frac{|3.1031 - \pi|}{\pi} \times 100\%$$
$$= 1.23\%$$

Using upper bound:

$$\text{Percentage error} = \frac{|3.1895 - \pi|}{\pi} \times 100\%$$
$$= 1.52\%$$

The largest possible percentage error is 1.52%

10 $2.05 \le u < 2.15,\ 15.5 \le v < 16.5,\ 9.805 \le a < 9.815$

$$t = \frac{v-u}{a}$$

$$\frac{15.5-2.15}{9.815} < t < \frac{16.5-2.05}{9.805}$$

$$1.360 < t < 1.474$$

11 Wall dimensions x and y have $3.35 \leqslant x < 3.45,\ 5.15 \leqslant y < 5.25$

Wall area A has $3.35 \times 5.15 \leqslant A < 3.45 \times 5.25$

$$17.2525 \leqslant A < 18.1125$$

Paint per square metre is q where $2.35 \leqslant q < 2.45$

Paint required is $v = Aq$ so $17.2525 \times 2.35 \leqslant v < 18.1125 \times 2.45$

$$42.2686 < v < 42.5644$$

Paint can contains V where $4.95 \leqslant V < 5.05$

Number of cans required is $N = \frac{v}{V}$

$$\frac{42.2686}{5.05} < N < \frac{42.5644}{4.95}$$

$$8.37 < N < 8.60$$

Jake will need to by 9 cans.

12 a $8+3 \leqslant p+q \leqslant 10+7$. Calculation by central value is $p+q = 9+5 = 14$

$$\text{Percentage error} = \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$

The percentage error will be the same at numerator no matter whether max or min value is used, but will be greater overall with the min value as denominator.

$$\text{Maximum percentage error} = \frac{|14-11|}{11} \times 100\%$$
$$= 27.2\%$$

b $8-7 \leqslant p-q \leqslant 10-3$; central value is $9-5=4$

$$\text{Percentage error} = \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$

The percentage error will be the same at numerator no matter whether max or min value is used, but will be greater overall with the min value as denominator.

$$\text{Maximum percentage error} = \frac{|4-1|}{1} \times 100\%$$
$$= 300\%$$

13 a i $p(0) = -0.5$

ii $p(50) = 0.149$

iii $p(100) = 1.22$

b (i) and (iii) are not possible as probabilities, since a probability can only take values in the interval $0 \leqslant p \leqslant 1$

14 a Using GDC:

$n = 10 \times 12$

$I\% = 5\%$

$PV = 1000$

$PMT = 0$

$FV = ?$

$P/Y = 12$

$C/Y = 12$

Solving for FV: $FV = -1647.01$

The true account balance at the end of 10 years is \$1647.01 so total interest is \$647.01

$$\text{Percentage error} = \left|\frac{\text{approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$
$$= \frac{|500 - 647.01|}{647.01} \times 100\%$$
$$= 22.7\%$$

b

$n = 10 \times 12$

$I\% = 5\%$

$PV = 1000$

$PMT = ?$

$FV = -500$

$P/Y = 12$

$C/Y = 12$

Solving for PMT: $PMT = -7.39$

She could withdraw \$7.39 each month and still have \$500 at the end.

15 a For the first two years:

$n = 2 \times 12$

$I\% = 2\%$

$PV = 24000$

$PMT = -300$

$FV = ?$

$P/Y = 12$

$C/Y = 12$

Solving for FV: $FV = -17638.93$

A balance of ¥17 638.93 is carried over into the higher interest period.

For the subsequent period:

$n = ?$

$I\% = 8\%$

$PV = 17638.93$

$PMT = -300$

$FV = 0$

$P/Y = 12$

$C/Y = 12$

Solving for n: $n = 74.8$

In total, Kanmin will need to pay for $24 + 75 = 99$ months

b

$n = 75$

$I\% = 8\%$

$PV = 17638.93$

$PMT = -300$

$FV = ?$

$P/Y = 12$

$C/Y = 12$

Solving for FV: $FV = 35.99$

In the final month, ¥300 would overpay by ¥35.99 so the final payment is actually only ¥264.01

16 Option A:

$n = 4 \times 25$

$I\% = 5\%$

$PV = 200000$

$PMT = ?$

$FV = 0$

$P/Y = 4$

$C/Y = 4$

Solving for PMT: $PMT = -3514.86$

Total amount paid: $3514.86 \times 100 = £351\,486$

Option B:

$n = 12 \times 30$

$I\% = 4\%$

$PV = 200000$

$PMT = ?$

$FV = 0$

$P/Y = 12$

$C/Y = 12$

Solving for PMT: $PMT = -954.83$

Total amount paid: $954.83 \times 360 = £343\,738.80$

Option B Is preferred, if Orlaigh wishes to minimise the total amount repaid.

17 a

$n = 12 \times 20$

$I\% = 4.8\%$

$PV = 100000$

$PMT = ?$

$FV = 0$

$P/Y = 12$

$C/Y = 12$

Solving for PMT: $PMT = -648.96$

Monthly payment is \$648.96

b

$n = ?$

$I\% = 4.8\%$

$PV = 100000$

$PMT = 748.96$

$FV = 0$

$P/Y = 12$

$C/Y = 12$

Solving for n: $n = 191.3$

Chad would need to pay for 192 months

$n = 192$

$I\% = 4.8\%$

$PV = 100000$

$PMT = 748.96$

$FV = ?$

$P/Y = 12$

$C/Y = 12$

Solving for FV: $FV = 513.71$

So in the final month, he would only pay $748.96 - 513.71 = 235.25$

In total, he would pay $192 \times 748.96 - 513.71 = \$143\,286.61$

If he only paid the minimum each month he would pay $240 \times 648.96 = \$155\,750.40$

The difference is $\$12\,463.79 \approx \$12\,500$

18 a

$n = 12 \times 10$

$I\% = 6\%$

$PV = 6000$

$PMT = ?$

$FV = 0$

$P/Y = 12$

$C/Y = 12$

Solving for PMT: $PMT = -66.61$

Monthly payment is €66.61

b Total paid $= 120 \times 66.61 = €7993.20$

Percentage of the total repayment that is interest:

$$\left(\frac{1993.20}{6000}\right) \times 100\% = 33.2\%$$

c

$n = 12 \times 5$

$I\% = 6\%$

$PV = 6000$

$PMT = 66.61$

$FV = ?$

$P/Y = 12$

$C/Y = 12$

Solving for FV: $FV = -3445.56$

So after 5 years, a total of $60 \times 66.61 = €3996.60$ has been paid, and the loan outstanding is €3445.56

Of the payments, $6000 - 3445.56 = €2554.44$ has been paying capital, and the remaining $3996.60 - 2554.44 = 1442.16$ has been paying interest.

The percentage of total interest paid this represents is:

$$\left(\frac{1442.16}{1993.20}\right) \times 100\% = 72.4\%$$

19 $-5.5 < y \leqslant -4.5,\ 15 \leqslant x < 25$

Then the maximum bound for $(x - y)(x + y) = x^2 - y^2$ is $25^2 — 4.52 = 604.75$

Tip: There is a temptation to work out the maximum for $x - y$ and a maximum for $x + y$ and take their product, but since they are not independent, those maximum values cannot be achieved simultaneously, so this is not a valid method.

20 **a** $2.55 \leqslant a < 2.65$

b $10^{2.55} \leqslant 10^a < 10^{2.65}$

$354.8 \leqslant 10^a < 446.7$

c The interval of possible values is wholly contained within the statement $10^a = 400$ (to 1 s.f.).

Worked solutions

12 Applications and interpretation: Solving equations with technology

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 12A

7 **a** Let b be number of boys and g be number of girls.

$$\begin{cases} b + g = 60 \\ 2b - g = 0 \end{cases}$$

b From calculator solver: $g = 40$

8 **a** Let w be the price of a widget and g be the price of a gizmo, each in cents.

$$\begin{cases} 3w + 7g = 1410 \\ 5w + 4g = 1200 \end{cases}$$

b From calculator solver: $w = 120, g = 150$

$w + g = \$2.70$

9 $u_6 = 16 = a + 5d$

$$S_{10} = 169 = \frac{10}{2}(2a + 9d)$$

$$\begin{cases} a + 5d = 16 \\ 10a + 45d = 169 \end{cases}$$

From calculator solver: $a = 25, d = -1.8$

10

$$S_5 = 19 = \frac{5}{2}(2a + 4d)$$

$$S_{11} = 48.4 = \frac{11}{2}(2a + 10d)$$

$$\begin{cases} 5a + 10d = 19 \\ 11a + 55d = 48.4 \end{cases}$$

From calculator solver: $a = 3.4, d = 0.2$

11 Let t be the price of a ticket, d be the price of a drink and p be the price of popcorn, each in dollars.

$$\begin{cases} 9t + 6d + 4p = 150 \\ 6t + 4d + 3p = 102 \\ 5t + 2d + 2p = 75 \end{cases}$$

From calculator solver: $t = 10.5, d = 5.25, p = 6$

A ticket costs $10.50, a drink costs $5.25 and popcorn costs $6.

12 Let a be the amount (in pounds) invested in account A, b in B and c in C.

$$\begin{cases} a+b+c=10000 \\ 0.02a+0.03b+0.04c=286 \\ a-b=300 \end{cases}$$

From calculator solver: $a=3900, b=3600, c=2500$

He invested \$3900 in account A, \$3600 in account B and £2 500 in account C.

13 Let c be the number of chocolate cakes, f the number of fruit cakes and s the number of sponge cakes.

$$\begin{cases} 225c+320f+400s=3300 \\ 300c+250f+325s=3100 \\ 12c+10f+8s=114 \end{cases}$$

From calculator solver: $c=4, f=5, s=2$

14 Let c be the price of a cow, s the price of a sheep and p the price of a pig (all in £).

$$\begin{cases} 3c+11s-32p=136 \\ c-7s=0 \\ c-8p=0 \end{cases}$$

From calculator solver: $c=238, s=34, p=29.75$

So the cost of a cow, a sheep and a pig is $238+34+29.75=£301.75$

15 Let r be the number of red cars, b the number of blue cars and s the number of grey cars.

$$\begin{cases} r+b+g=\frac{360}{192}g \\ -r+b=2 \\ r+b-g=-2 \end{cases}$$

From calculator solver: $r=6, b=8, g=16$

$$P(R)=\frac{6}{30}=0.2$$

16 Let a be the first digit, b the second digit and c the third digit so that the number is $100a+10b+c$, and the number reversed has value $100c+10b+a$

$$\begin{cases} a+b+c=16 \\ 2a-2b+c=0 \text{ or } 2a-2b-c=0 \\ 99a-99c=297 \end{cases}$$

From calculator solver: $a=5.9, b=7.3, c=2.9$ (reject!)or $a=7, b=5, c=4$

The only valid solution is 754

17

$$\begin{cases} x^2+y^2+z^2=26 \\ x^2+2y^2+3z^2=67 \\ x^2-y^2+z^2=8 \end{cases}$$

From calculator solver: $x^2=1, y^2=9, z^2=16$

So $x=\pm1, y=\pm3, z=\pm4$

18 Let the three values be a, b and c, where $a \leqslant b \leqslant c$

Then the mean is $\frac{a+b+c}{3}$ and the median is b; the range is $c - a$

$$\begin{cases} \dfrac{a+b+c}{3} = 2b \\ c - a = 5b \\ b - a = 1 \end{cases}$$

Simplifying and rearranging to a consistent order:

$$\begin{cases} a - 5b + c = 0 \\ a + 5b - c = 0 \\ a - b = -1 \end{cases}$$

From calculator solver: $a = 0, b = 1, c = 5$

The greatest of the three numbers is 5.

Exercise 12B

10 Let the base be x so that the height is $x - 2$

$$\frac{x(x-2)}{2} = 40$$

$$x^2 - 2x - 80 = 0$$

From calculator solver: $x = 10$ (rejecting the negative solution).

The height is $x - 2 = 8$ cm.

11

$$u_1 = 6$$
$$S_3 = u_1(1 + r + r^2) = 58.5$$
$$r^2 + r + 1 - \frac{58.5}{6} = 0$$

From calculator solver: $r = 2.5$ or -3.5

12

$$u_1 = 3.4$$
$$S_4 = \frac{u_1(r^4 - 1)}{r - 1} = 51$$
$$r^4 - 1 = \frac{51(r-1)}{3.4}$$
$$r^4 - 15r + 14 = 0$$

From calculator solver: $r = 2$

13 **a**

$$2x - 1 = \frac{5}{x+3}$$
$$(2x - 1)(x + 3) = 5$$
$$2x^2 \mp 5x - 3 = 5$$
$$2x^2 + 5x - 8 = 0$$

b From calculator solver: $x = -3.61$ or 1.11

14 a $x^2 - 3 - x^{-1} = 0$

Multiplying through by x:

$$x^3 - 3x - 1 = 0$$

b From calculator solver: $x = 1.88, -1.53, -0.347$

15 a By Pythagoras, $x^2 + (x + 5)^2 = (2x - 1)^2$

$$\begin{aligned} x^2 + x^2 + 10x + 25 &= 4x^2 - 4x + 1 \\ 2x^2 - 14x - 24 &= 0 \\ x^2 - 7x - 12 &= 0 \end{aligned}$$

b From calculator solver, $x = 8.42, -1.42$

c From context, $x > 0$ (length must be positive) so reject the negative solution.

$$x = 8.42$$

16 a

$$\begin{aligned} x(4x + 3) + x(x - 1) &= 19.8 \\ 5x^2 + 2x - 19.8 &= 0 \end{aligned}$$

b From calculator solver: $x = 1.8$ or -2.2 (reject)

$$x = 1.8 \text{ m}$$

17 After the squares are cut out, the remaining side lengths of the base are $50 - 2x$ and $40 - 2x$.

$$\begin{aligned} V = (50 - 2x)(40 - 2x)x &= 6000 \\ 4x^3 - 180x^2 + 2000x - 6000 &= 0 \end{aligned}$$

From calculator solver: $x = 5, 10, 30$

From context, $x < 20$ so two possible solutions are $x = 5$ or $x = 10$

18 a

$$V = \pi r^2 h = 600 \qquad (1)$$

$$S = 2\pi r^2 + 2\pi rh = 450 \qquad (2)$$

Rearranging (2): $\pi r(r + h) = 225$

$$h = \frac{225}{\pi r} - r = \frac{225 - \pi r^2}{\pi r}$$

b Substituting into (1):

$$\begin{aligned} r(225 - \pi r^2) &= 600 \\ \pi r^3 - 225r + 600 &= 0 \end{aligned}$$

c From calculator solver: $r = 6.50, 3.07$ or -9.57

From context, $r > 0$ so the possible radius length is 6.50 cm or 3.07 cm.

19 Let x be the edge length of the cube.

$$x^3 + 6x^2 - 160 = 0$$

From calculator solver: $x = 4$

20

$$u_1 = 1, d = 6$$
$$S_n = \frac{n}{2}\big(2u_1 + d(n-1)\big) = 1160$$
$$3n^2 - 2n - 1160 = 0$$

From calculator solver: $n = 20$ or -19.3 (reject)

$n = 20$

21 Let x be the lower of the base values.

$$(x+1)^3 - x^3 = 61$$
$$3x^2 + 3x - 60 = 0$$

From calculator solver: $x = 4$ or -5

So the values are either 4 and 5 or -5 and -4

22 **a** Each vertex has a directed diagonal to each non-adjacent vertex. Not counting the vertex itself or either of its immediate neighbours, there must be $n - 3$ non-adjacent vertices for each vertex.

So with n vertices, there are $n(n-3)$ directed diagonals.

Each diagonal would be counted twice (both directions) so there are in total $\frac{1}{2}n(n-3)$ diagonals.

b

$$\frac{n(n-3)}{2} = 35$$
$$n^2 - 3n - 70 = 0$$

From calculator solver: $n = -7$ or 10

From context, $n > 0$ so $n = 10$.

Mixed Practice

1 Let the height be x so the base is $x + 6$.

$$\frac{x(x+6)}{2} = 20$$
$$x^2 + 6x - 40 = 0$$

From calculator solver: $x = 4$ or -10

Rejecting the negative solution (length must be positive), the height must be 4 cm.

2 **a**

$$(2x+5)(x-4) = 10$$
$$2x^2 - 3x - 20 = 10$$
$$2x^2 - 3x - 30 = 0$$

b From calculator solver: $x = 4.69, -3.19$

Worked solutions

3 **a**

$$x^2 - 7 = \frac{2}{x+4}$$
$$(x^2 - 7)(x + 4) = 2$$
$$x^3 + 4x^2 - 7x - 30 = 0$$

b From calculator solver: $x = 2.70, -3, -3.70$

4 **a**

$$x(x + 1) = 10$$
$$x^2 + x - 10 = 0$$

b From calculator solver: $x = 2.70, -3.70$

5 Let x be the length of the shorter sides.

$$x^2(x + 2) = 10$$
$$x^3 + 2x^2 - 10 = 0$$

From calculator solver: $x = 1.65$ cm

6 **a** Let f be the number of flies and s the number of spiders.

$$\begin{cases} 6f + 8s = 142 \\ f + s = 20 \end{cases}$$

b Using calculator solver: $f = 9, s = 11$

There are 11 spiders.

7 Let p be the cost of a pail, v the cost of vinegar and brown paper (in dollars)

$$\begin{cases} 7p + 5v = 70 \\ 5p + 7v = 74 \end{cases}$$

Using calculator solver: $p = 5, v = 7$

Vinegar and brown paper is $2 more expensive than a pail of water.

8 Let s be the number of science books Daniel buys and a be the number of art books Daniel buys.

$$\begin{cases} 8s + 12a = 168 \\ 4s + 24a = 264 \end{cases}$$

Using calculator solver: $s = 6, a = 10$

Daniel bought 6 science and 10 art, Alessia bought 3 science and 20 art.

Altogether they bought 39 books.

9 **a** $x + y = 10\,000$

b $2(12) + 3(5) = 39$ AUD

c $12x + 5y = 108\,800$

d Using calculator solver: $x = 8400, y = 1600$

10

$$\begin{cases} u_{40} = u_1 + 39d = 106 \\ S_{40} = 20(u_1 + 106) = 1900 \end{cases}$$

Simplifying and rearranging:

$$\begin{cases} u_1 + 39d = 106 \\ 20u_1 = -220 \end{cases}$$

So $u_1 = -11$ and $d = \frac{106-u_1}{39} = 3$

11 a

$$\begin{aligned} A &= (x+1)^2 - 1 \\ &= x^2 + 2x \end{aligned}$$

b $x^2 + 2x - 109.25 = 0$

Using calculator solver: $x = 9.5 \text{ or } -11.5$

Rejecting the negative solution, $x = 9.5$

c Fence length: $4(x+1) = 42$ m

12 Let x be the middle of the three consecutive numbers, so the others are $x-1$ and $x+1$ and the mean is x.

$$\begin{aligned} x(x-1)(x+1) &= 504 \\ x^3 - x - 504 &= 0 \end{aligned}$$

Using calculator solver: $x = 8$

The mean of the numbers is 8.

13

$$\begin{aligned} u_1 &= 8 \\ S_3 &= u_1(1 + r + r^2) = 14 \\ r^2 + r + 1 &= \frac{14}{8} \\ r^2 + r - \frac{6}{8} &= 0 \end{aligned}$$

Using calculator solver: $r = 0.5 \text{ or } -1.5$

14

$$\begin{aligned} u_1 &= 2 \\ S_4 &= \frac{u_1(r^4 - 1)}{r - 1} = -40 \\ r^4 - 1 &= -20(r-1) \\ r^4 + 20r - 21 &= 0 \end{aligned}$$

Using calculator solver: $r = -3$

15 a Using cosine rule:

$$\begin{aligned} (x+2)^2 &= x^2 + (2x)^2 - 2x(2x)\cos 60° \\ x^2 + 4x + 4 &= 5x^2 - 2x^2 \\ 2x^2 - 4x - 4 &= 0 \\ x^2 - 2x - 2 &= 0 \end{aligned}$$

b From calculator solver, the only positive solution to this is $x = 2.73$

16 $u_1 = 100, d = -2$

$$S_n = \frac{n}{2}\left(2u_1 + d(n-1)\right) = 2440$$

$$n\left(100 - (n-1)\right) = 2440$$
$$n^2 - 101n + 2440 = 0$$

Using calculator solver, $n = 40$ or 61

17 a

$$V = x(x+2)(2x+3)$$
$$= 2x^3 + 7x^2 + 6x$$

b $2x^3 + 7x^2 + 6x - 31.5 = 0$

Using calculator solver: $x = 1.5$

The tank dimensions are 1.5 m × 3.5 m × 6 m

18 a Arithmetic sequence: $u_1 = 1, d = 1$

$$S_{20} = \frac{20}{2}\left(2 \times 1 + 19(1)\right) = 210$$

b

$$\frac{n}{2}\left(2 \times 1 + 1(n-1)\right) = 3240$$
$$n^2 + n - 6480 = 0$$

Using calculator solver: The only positive integer solution is $n = 80$

c i

$$\frac{n}{2}(2 + n - 1) = S$$
$$n^2 + n - 2S = 0$$

ii $n^2 + n - 4200 = 0$

Using calculator solver: $n = 64.3$ or -65.3

There is no positive integer solution for n, so the cans cannot form an exact triangle. A 64 row triangle could be made, with some spare cans, but there are not enough for a 65 row triangle.

19 Let s be the number who learn Spanish, f be the number who learn French and b be the number who learn both languages.

$$\begin{cases} s + f + b = 30 \\ s - f = 1 \\ s + f - b = 12 \end{cases}$$

From calculator solver: $s = 11, f = 10, b = 9$

$$\text{P(learn both)} = \frac{b}{30} = 0.3$$

20 $\frac{dy}{dx} = 2ax$

$$\frac{dy}{dx}(1) = 2a + b = 12$$

$$y(1) = a + b = 7$$

$$\begin{cases} 2a + b = 12 \\ a + b = 7 \end{cases}$$

Using calculator solver: $a = 5, b = 2$

$$y(2) = 5(2^2) + 2(2) = 24$$

21 Let x be the edge length of the cube

$$x^3 + 6x^2 = 100$$

Using calculator solver: $x = 3.28$

22 Let x be the lesser value, so the greater value is $x + 1$

$$(x + 1)^3 - x^3 = 2$$
$$3x^2 + 3x - 1 = 0$$

Using calculator solver: $x = 0.264 \text{ or } -1.26$

23

$$\begin{cases} x^3 + y^3 + z^3 = 8 \\ x^3 + 2y^3 + 3z^3 = 23 \\ x^3 - y^3 + 2z^3 = 18 \end{cases}$$

Using calculator solver: $x^3 = 1, y^3 = -1, z^3 - 8$

So $x = 1, y = -1, z = 2$

24 Each person shakes hands with each other person, so the total number of directed handshakes is $n(n - 1)$.

Since this counts each shake twice (A shakes B is the same as B shakes A), the total number of (undirected) shakes is $\frac{n(n-1)}{2}$

$$\frac{n(n - 1)}{2} = 190$$
$$n^2 - n - 380 = 0$$

Using calculator solver: The only positive integer $n = 20$.

13 Applications and interpretation: Mathematical models

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 13A

7 a i For every year after purchase, the car will travel 8200 miles.

ii The car had travelled 29 400 miles when purchased.

b $M(5) = 41 + 29.4 = 70.4$

After 5 years, the car is predicted to have travelled 70 400 miles.

c

$$8.2y + 29.4 > 100$$
$$y > \frac{100 - 29.4}{8.2} = 8.6$$

The car will exceed 100 000 miles in the ninth year.

8 a $C = 1.25d + 2.5$

b $C(3.2) = 6.5$

A journey of 3.2 km would cost $6.50

c $C(6.5) = 10.63$

Ryan does not have enough money; such a journey would cost $10.63.

9 a i $k = 100$

ii

$$C(12.5) = 0 = 12.5m + 100$$
$$m = -\frac{100}{12.5} = -8$$

b $0 \leqslant t \leqslant 12.5$

10 a $C = md + k$

$$\begin{cases} C(0.8) = 630 = 0.8m + k \\ C(3) = 520 = 3m + k \end{cases}$$

Using calculator solver: $m = -50, k = 670$

$$C = -50d + 670$$

b $C(4.8) = 430$

The model predicts a house 4.8 miles from the centre would cost £430 000.

11 a

$$\begin{cases} w(2) = 4.08 = 2a + b \\ w(6) = 4.98 = 6a + b \end{cases}$$

Using calculator solver: $a = 0.225, b = 3.63$

b i a is the change in weight per week

ii b is the birth weight

12 a $10 + 0.12 \times 250 = 40$

$$C = \begin{cases} 0.12e + 10 & \text{for } 0 \leqslant e \leqslant 250 \\ 40 + 0.15(e - 250) & \text{for } e > 250 \end{cases}$$

Simplifying:

$$C = \begin{cases} 0.12e + 10 & \text{for } 0 \leqslant e \leqslant 250 \\ 0.15e + 2.5 & \text{for } e > 250 \end{cases}$$

b $C(300) = 0.15(300) + 2.5 = 47.5$

The cost for 300 kWh is £47.50

13 a i $C_H = 7.5m + 80$

ii $C_T = 5m + 125$

b The distance at which the two companies cost the same is found when $C_H = C_T$:

$$80 + 7.5m = 125 + 5m$$

$$2.5m = 45$$

$$m = 18$$

For distances above 18 miles, Tortoise Removals is cheaper.

14 a

$$\begin{aligned} D(p) &= a - 15p \\ D(75) &= 0 = a - 15 \times 75 \\ a &= 1125 \\ D(p) &= 1125 - 15p \end{aligned}$$

b

$$\begin{aligned} S(p) &= b + 12p \\ S(30) &= 0 = b + 12 \times 30 \\ b &= -360 \\ S(p) &= 12p - 360 \end{aligned}$$

c

$$\begin{aligned} C = S&: 1125 - 15p = 12p - 360 \\ 27p &= 765 \\ p &= 28.33 \end{aligned}$$

The equilibrium price is \$28.33

15 a $d(0) = 1.5$ m

b $d(20) = 2.3 = 20a + b$

$d(25) = 3.2 = 25a + b$

Using calculator solver: (i) $a = 0.18$, (ii) $b = -1.3$

c $d(15) = 2.1$ m

d

$d(x) = 2.5 = 0.18x - 1.3$

$$x = \frac{3.8}{0.18} = 21.1 \text{ m}$$

16 a Tax on £50 000 is $0.2 \times 37.5 = 7.5$

Tax on £150 000 is $7.5 + 0.4 \times 100 = 47.5$

$$T = \begin{cases} 0 & 0 \leqslant I \leqslant 12.5 \\ 0.2(I - 12.5) & 12.5 < I \leqslant 50 \\ 0.4(I - 50) + 7.5 & 50 < I \leqslant 150 \\ 0.45(I - 150) + 47.5 & I > 150 \end{cases}$$

Simplifying:

$$T = \begin{cases} 0 & 0 \leqslant I \leqslant 12.5 \\ 0.2I - 2.5 & 12.5 < I \leqslant 50 \\ 0.4I - 12.5 & 50 < I \leqslant 150 \\ 0.45I - 20 & I > 150 \end{cases}$$

b i $T(35) = 4.5$ so tax payable on £35 000 is £4 500

ii $T(70) = 15.5$ so tax payable on £70 000 is £15 500

c Since $50 > T(150) = 47.5$, this will fall in the final income bracket.

$T(I) = 50 = 0.45I - 20$

$$I = \frac{70}{0.45} = 155.556$$

An income of £155 556 would cause a payable tax of £50 000.

Exercise 13B

7 a Axis of symmetry lies midway between roots.

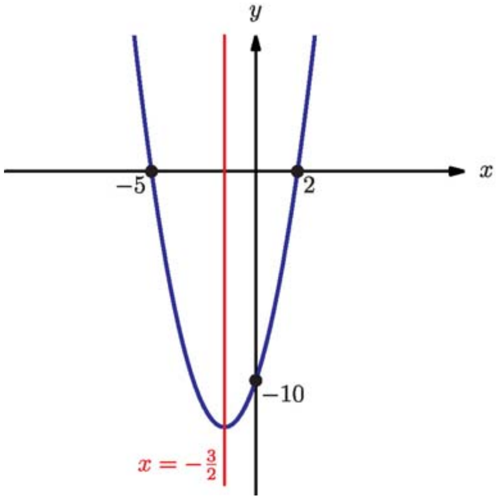
Q is $(-5,0)$

b

$$\begin{aligned} y &= (x + 5)(x - 2) \\ &= x^2 + 3x - 10 \end{aligned}$$

$b = 3, c = -10$

c



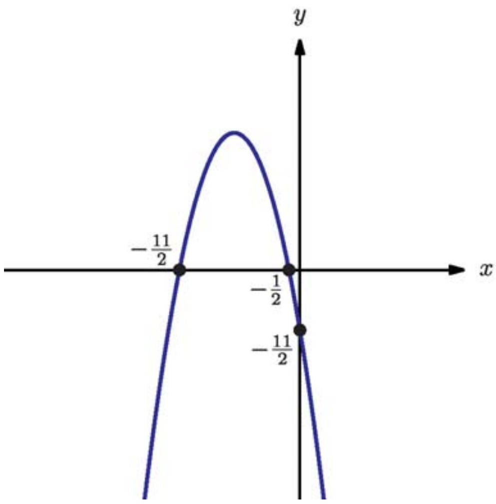
8 a x-coordinate of vertex lies midway between zeroes: $x = -3$

b

$$30.25a - 5.5b + c = 0$$
$$0.25a - 0.5b + c = 0$$
$$9a - 3b + c = 12.5$$

c Using calculator solver: $a = -2, b = -12, c = -5.5$

d



9 a

$$\begin{cases} 2500a + 50b + c = 1350 \\ 40\,000a + 200b + c = 5100 \\ 160\,000a + 400b + c = 3100 \end{cases}$$

b Using calculator solver: $a = -0.1, b = 50, c = -900$

c c is the profit the business is projected to make if it produces no items (that is, it would make a £900 loss).

d The maximum of the model is $(250, 5350)$.

i The model projects maximum profit when 250 items are produced.

ii The model projects maximum profit of £5350.

10 Let y be the height of the surface (in cm) and x the distance from the left edge of the road (in m).

Model with a parabola with roots at 0 and 7.5, passing through the point $(1,5)$

$$y = ax(7.5 - x)$$
$$y(1) = 6.5a = 5$$
$$a = 0.769$$

The maximum lies at the midpoint of the parabola: $y(3.75) = 0.769(3.75)^2 = 10.8$

The maximum height of the road is 10.8 cm

11 a $c = 1.45$

b i

$$-\frac{b}{2a} = 1.4$$
$$h(2.9) = 0 = 8.41a + 2.9b + 1.45$$

ii

$$\begin{cases} 2.8a + b = 0 \\ 8.41a + 2.9b + 1.45 = 0 \end{cases}$$

Using calculator solver: $a = -5, b = 14$

c $h(t) = -5t^2 + 14t + 1.45 = 5$

From calculator, $t = 0.282$ or 2.52

The ball is 5 m off the ground 0.282 seconds and 2.52 seconds after being thrown.

12 a $R = at + b$ where R is the radius in metres and d is the number of days that have passed since the first measurement.

$$R(0) = \sqrt{\frac{2}{\pi}} = b \text{ so } b = 0.798$$

$$R(4) = \sqrt{\frac{3}{\pi}} = 4a + b \text{ so } a = \frac{1}{4}\left(\sqrt{\frac{3}{\pi}} - \sqrt{\frac{2}{\pi}}\right) = 0.0448$$

The model projects $R(7) = 1.11$

So the area predicted after 7 days is $\pi(1.11)^2 = 3.88 \text{ m}^2$

b The model assumes that R increases at a constant rate, ignoring other features which might fluctuate (such as temperature, nutrients, space restrictions)

c In the model the algae would increase in area without limit; in reality, the algal growth would be limited by the dimensions of the pond.

Exercise 13C

9 a $N(0) = 4.68$ billion people have a mobile phone.

b 2%

c $N(6) = 4.68 \times 1.02^6 = 5.27$

In 2025, the model predicts 5.27 billion people will have a mobile phone.

10 a $M(0) = 1.5 \text{ mg l}^{-1}$

b $M(5) = 0.556 \text{ mg l}^{-1}$

c

$$M(t) = 0.25 = 1.5 \times 0.82^t$$
$$0.82^t = \frac{0.25}{1.5}$$
$$t = \frac{\log\left(\frac{0.25}{1.5}\right)}{\log 0.82} = 9.0287$$

$0.0287 \times 60 = 1.7 \approx 2$

The concentration reaches 0.25 mg l^{-1} after about 9 hours, 2 minutes

11 a

$$N(0) = 2.5 = k$$
$$N(5) = 7 = ke^{5r}$$
$$r = \frac{1}{5}\ln\left(\frac{7}{k}\right) = 0.206$$

b $N(30) = 1\,204.7$

After 30 minutes, the model predicts approximately 1 205 000 bacteria.

12 a

$$m(3) = 29.6 = ka^{-3} \qquad (1)$$
$$m(10) = 11.1 = ka^{-10} \qquad (2)$$

$$(1) \div (2): \frac{29.6}{11.1} = a^7$$
$$a = 1.15$$
$$k = 29.6a^3 = 45.1$$

b

$$m(t) = 1 = ka^{-t}$$
$$t = -\frac{\log(k^{-1})}{\log a} = 27.2 \text{ s}$$

13 a

$$V = 8500 \times \left(\frac{6800}{8500}\right)^t$$
$$= 8500 \times 0.8^t$$

b $V(10) = 912.68$

After 10 years the model predicts a value of \$912.68

c $d = 500$

Then $V(0) = 8500 = c + d$ so $c = 8000$

14 a

$$m(15) = \frac{1}{2}m(0)$$
$$ke^{-15c} = \frac{1}{2}k$$
$$c = -\frac{1}{15}\ln\left(\frac{1}{2}\right) = 0.0462$$

b

$$m(t) = 0.01k$$
$$e^{-ct} = 0.01$$
$$t = -\frac{1}{c}\ln 0.01 = 99.7 \text{ hours}$$

15 $v \to 50$ as $t \to \infty$

The model predicts the speed rises towards a limit of 50 m s^{-1}

16 a The soup is cooling towards room temperature 20° C so $c = 20$

$T(0) = 85 = k + c$ so $k = 65$

$T(5) = 20 + 0.75(65) = 68.75 = ka^5 + 20$

$a = (0.75)^{\frac{1}{5}} = 0.944$

$T(8) = 65 \times 0.944^8 + 20 = 61.0\,°\text{C}$

b i k represents the temperature difference between the soup and the room at $t = 0$ so would be unchanged.

ii a represents the ratio of decay in temperature difference each minute. The value of a would still be between 0 and 1 (the soup will still cool) but will be a greater value (the soup will cool more slowly).

Exercise 13D

9 a $F = kx$

$12 = 5k$ so $k = 2.4$

$F = 2.4x$

b i $F(4) = 9.6$ N

ii $F(x) = 20 = 2.4x$

$x = 8.3$ cm

10 a $T = k\sqrt{l}$

$1 = k\sqrt{25}$ so $k = 0.2$

$T = 0.2\sqrt{l}$

b i $T(60) = 0.2\sqrt{60} = 1.55$ s

ii $T(l) = 0.8 = 0.2\sqrt{l}$

$l = 16$ cm

11 a

$$\begin{aligned} P &= ks^3 \\ 2.05 \times 10^6 &= k \times 10^3 \\ k &= 2.05 \times 10^3 \\ P(14) &= 2.05 \times 10^3 \times 14^3 = 5.63 \times 10^6 \text{ Watts} \end{aligned}$$

b

$$P(s) = 10^7 = 2.05 \times 10^3 \times s^3$$

$$s = \sqrt[3]{\frac{10^4}{2.05}} = 17.0 \text{ m s}^{-1}$$

12 a

$$\begin{aligned} P &= \frac{k}{V} \\ 45000 &= \frac{k}{20} \\ k &= 9 \times 10^5 \\ P &= \frac{900\,000}{V} \end{aligned}$$

b i $P(125) = 7200$ Pa

ii

$$\begin{aligned} P(V) = 80000 &= \frac{9 \times 10^5}{V} \\ V &= \frac{90}{8} = 11.25 \text{ cm}^3 \end{aligned}$$

13

$$\begin{aligned} J &= \frac{k}{T} \\ 55 &= \frac{k}{8} \\ k &= 440 \\ J(15) &= \frac{440}{15} = 29.3 \end{aligned}$$

The model predicts sales of 29 sweatshirts.

14 a

$$\begin{cases} -8a + 4b - 2c + d = -1 \\ -a + b - c + d = -8 \\ a + b + c + d = 2 \\ 8a + b + c + d = -5 \end{cases}$$

b Using calculator solver: $a = -2, b = 0, c = 7, d = -3$

15 a $T = ad^3 + bd^2 + cd + k$

$$\begin{cases} a(0.723)^3 + b(0.723)^3 + c(0.723) + k = 0.615 \\ a(1.52)^3 + b(1.52)^3 + c(1.52) + k = 1.88 \\ a(9.54)^3 + b(9.54)^3 + c(9.54) + k = 29.5 \\ a(30.1)^3 + b(30.1)^3 + c(30.1) + k = 165 \end{cases}$$

Using calculator solver: $a = -0.00342, b = 0.251, c = 1.04, k = -0.265$

b **i** The model predicts that when $d = 5.20$, $T = 11.4$ years

ii

$$\text{Percentage error} = \frac{|\text{true value} - \text{approximate value}|}{\text{true value}} \times 100\%$$
$$= \frac{|11.9 - 11.4|}{11.9} \times 100\%$$
$$= 3.88\%$$

c The maximum of this model for positive d is $(50.9, 252)$ so the maximum orbital period predicted is 252 years.

16

$$y = k_1 z^3$$
$$x = \frac{k_2}{\sqrt{y}} = \frac{k_2}{\sqrt{k_1 z^3}} = kz^{\frac{3}{2}}$$

17 If $F(d)$ is the force of attraction between the magnets when at distance d apart:

$$F(d) = \frac{k}{d^2}$$
$$F(1.1d) = \frac{k}{1.21d^2} = \frac{1}{1.21}F(d) = 0.826F(d)$$

The force of attraction decreases by 17.4%

18 $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$

$$\begin{cases} a + b + c + d = -2 \\ 8a + 4b + 2c + d = -4 \\ 27a + 9b + 3c + d = -2 \\ 64a + 16b + 4c + d = 10 \end{cases}$$

Using the calculator solver: $a = 1, b = -4, c = 3, d = -2$

$$\mathrm{f}(5) = 125 - 4(25) + 3(5) - 2 = 38$$

19 Let $T(m, n)$ be the time taken to build an estate of m houses with n builders.

$$T = \frac{km}{n}$$
$$T(7,7) = 7 = \frac{7k}{7}$$
$$k = 7$$

Then $T(2,2) = \dfrac{7 \times 2}{2} = 7$

It will still take 7 days.

20

Tip: This can be approached naively by setting up a general cubic and using a set of three simultaneous equations to reduce to only one parameter.

More swiftly, since the pattern is of three values for which $\mathrm{f}(x) = x$, we can approach by finding the function $\mathrm{g}(x) = \mathrm{f}(x) - x$, for which roots are known.

Method 1:

$\mathrm{f}(x) = ax^3 + bx^2 + cx + d$

$$\begin{cases} d = 0 (1) \\ a + b + c + d = 1 (2) \\ 8a + 4b + 2c + d = 2 (3) \end{cases}$$

Substituting:

$a + b + c = 1 (4)$

$8a + 4b + 2c = 2 (5)$

$(5) - 2(4): 6a + 2b = 0$

$b = -3a$

Then from (4): $c = 1 + 2a$

The general equation is $\mathrm{f}(x) = ax^3 - 3ax^2 + (1 + 2a)x$

Method 2:

Let $\mathrm{g}(x) = \mathrm{f}(x) - x$

Then $\mathrm{g}(0) = \mathrm{g}(1) = \mathrm{g}(2) = 0$

$\mathrm{g}(x)$ has roots at $0, 1$ and 2 so $\mathrm{g}(x) = kx(x-1)(x-2)$

Then $\mathrm{f}(x) = kx(x-1)(x-2) + x$

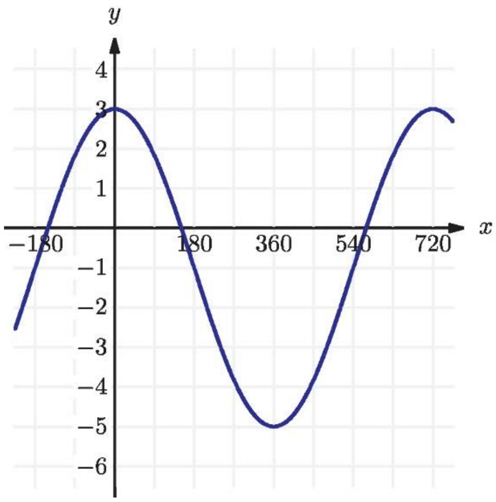
Exercise 13E

3 **a** Range is from -1 to 5 so amplitude is 3

b Period is 45 so $q = \frac{360}{45} = 8$

c Centre line is at $y = 2$ so $r = 2$

4



a Range is from -5 to 3 so amplitude is 4

b Period is 720 so $q = \frac{360}{720} = \frac{1}{2}$

c Centre line is at $y = -1$ so $r = -1$

5 **a** 50 m

b 1.8 m

c Wavelength $= \frac{360}{72} = 5$ m

6 **a** **i** Maximum height is $0.58 + 1.1 = 1.68$ m above ground.

ii Minimum height is $-0.58 + 1.1 = 0.52$ m above the ground.

b Half a period is 0.6 s so the full period is 1.2 s

$$b = \frac{360}{1.2} = 300° \text{ s}^{-1}$$

c $h(4) = 0.58\cos(1200) + 1.1 = 0.81$ m

7 **a** 152 cm

b

$$450t = 180$$
$$t = 0.4 \text{ s}$$

c Central height is 140 cm.

8 **a** The distance from the point can range from -40 to 40, and has period 20 s.

$d(0) = 0 = r$.

Then $p = 40,\ q = \frac{360}{20} = 18° \text{ s}^{-1}$

b $d(t) = -10$

From calculator, least positive solution for t is $t = 10.8$

The bike is first 10 m south of the diameter 10.8 s after starting.

9 Amplitude is $a = \pm 4.5$ °C

Period is $24 = \frac{360}{b}$ so $b = 15$

Central temperature is $d = 10.5$ °C

$$T = a\sin(15t) + 10.5$$

Require that the minimum occurs at $t = -6$

$$T(-6) = a\sin(-90°) + 10.5 = 6$$
$$a = 4.5$$
$$T(3) = 4.5\sin(45°) + 10.5 = 13.7\text{ °C}$$

10 Amplitude is $a = \frac{3.5}{2} = 1.75$ m

Period is $12 = \frac{360}{b}$ so $b = 30$

Central height is $d = 3.75$ m

$$h = 1.75\cos(30t) + 3.75$$
$$h(8.5) = 3.30\text{ m}$$

Exercise 13F

5 **a** $P = 67 \times 1.01^n$

b Growth rate is not necessarily going to be constant. Population changes due to birth, death and migration; events may change these elements.

Also, the model is not precise and even if the growth rate were to be constant, errors would accumulate over the years to 2100.

6 **a** Pressure and time will both be non-negative.

$$P \geqslant 0 \Rightarrow t \leqslant \frac{2.2}{0.04}$$

Domain: $0 \leqslant t \leqslant 55$

b An exponential model would show the rate of pressure decrease slowing as time passes, so that the pressure tends to zero but does not become negative.

7 **a** 0 °C

b Require $T(0) = 100$ and $T \to 20$ as $t \to \infty$

$$T = 20 + 80 \times 0.87^t$$

Mixed Practice

1 **a** $C(0) = 16\%$

b $C(30) = 52\%$

c

$$C(t) = 100 = 16 + 1.2t$$
$$t = \frac{84}{1.2} = 70\text{ minutes}$$

2 **a** $c = 3$

b

$$-\frac{b}{2a} = 6$$
$$f(6) = 36a + 6b + c = 15$$
$$36a + 6b = 12$$

c

$$\begin{cases} 12a + b = 0 \\ 36a + 6b = 12 \end{cases}$$

From calculator solver: $a = -\frac{1}{3}, b = 4$

3 **a** $f(0) = p + q = 5 \quad (1)$

$f(1) = 0.4p + q = 4.1 \quad (2)$

$(2) - (1): 0.6p = -0.9$

$p = -1.5, q = 3.5$

b $y = 3.5$

4 **a**

$$V = kT$$
$$140 = 350k$$
$$k = \frac{140}{350} = 0.4$$
$$V = 0.4T$$

b $V(293) = 0.4 \times 293 = 117\text{ cm}^3$

5 **a** $y = mt + n$

$y(1) = m + n = 150 \quad (1)$

$y(6) = 6m + n = 600 \quad (2)$

$(2) - (1): 5m = 450 \rightarrow m = 90$

b m represents the change in number of apartments each year.

c $n = 150 - m = 60$

d n is the initial number of apartments.

6 **a**

$$\begin{cases} -8p + 4q - 2r = -8 \\ p + q + r = -2 \\ 8p + 4q + 2r = 0 \end{cases}$$

b Using calculator solver: $p = 1, q = -1, r = -2$

7 **a** **i** Amplitude $= 3$

ii Period $= 180°$

b **i** Amplitude $a = \frac{1-(-3)}{2} = 2$

ii $b = \frac{360°}{\text{period}} = \frac{360°}{360°} = 1$

iii Centre value $c = -1$

c The graphs intersect 5 times in the domain $-180° \leqslant x \leqslant 360°$

8 **a** \$1.00

b \$2.00

c $350 \leqslant w < 500$

9 **a** $\text{f}(0) = 6 = 1 + b$ so $b = 5$

$\text{f}(1) = 9 = a + 5$ so $a = 4$

b $\text{f}(2) = 4^2 + 5 = 21$

c

$$\begin{aligned} \text{f}(c) &= 5.5 = 4^c + 5 \\ 4^c &= 0.5 \\ c &= -0.5 \end{aligned}$$

10 Let $F(d)$ be the gravitational force for distance d from the centre of the earth.

$$F(d) = \frac{k}{d^2}$$

The gravitational force on the satellite before launch is $F(6000) = \frac{k}{3.6 \times 10^7}$

The gravitational force on the satellite in orbit is $F(7500) = \frac{k}{5.625 \times 10^7}$

$$\begin{aligned} \text{Percentage decrease} &= \frac{\text{before} - \text{after}}{\text{before}} \times 100\% \\ &= \frac{\frac{1}{3.6} - \frac{1}{5.625}}{\frac{1}{3.6}} \times 100\% \\ &= 36\% \end{aligned}$$

11 **a** **i** $94 - 54 = 40°$ C

ii $54 - 34 = 20°$ C

iii $34 - 24 = 10°$ C

b The temperature lost halves each minute, so during the fourth minute, the temperature should fall 5° C so $k = 24 - 5 = 19$

c

d i

$$y = p(2^{-t}) + q$$
$$y(0) = p + q = 94$$

ii $y(1) = \frac{p}{2} + q = 54$

e $p = 80, q = 14$

f $y = 14$

g i $r = -0.878$

ii $y = 71.6 - 11.7t$

h Using this linear model, $y(3) = 36.7$

i

$$\text{Percentage error} = \frac{|\text{True value} - \text{approximate value}|}{\text{True value}} \times 100\%$$
$$= \frac{24 - 36.7}{24} \times 100\%$$
$$= 52.8\%$$

12 a $C(0) = 2.5 - 1 = 1.5$

b $C = 2.5$

c

$$C(t) = 2.4 = 2.5 - 2^{-t}$$
$$2^{-t} = 0.1$$
$$t = 3.3219$$

$0.3219 \times 60 = 19.3$

It takes 3 hours and 19 minutes to reach this charge.

13 a i Half a rotation: P is at the top, so 100 m above ground level.

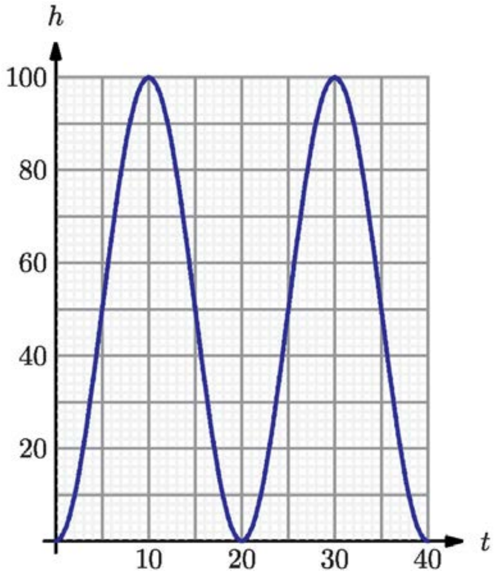
ii Three quarters of a rotation: P is at the rightmost point, so 50 m above ground level.

b i After 8 minutes, P is as far from the top as it is from the base after 2 minutes:

$h(8) = 100 - h(2) = 100 - 9.5 = 90.5$

ii A complete rotation takes 20 minutes so $h(21) = h(1) = 2.4$

c



d Initial position is $h(0) = 0$ so $a + c = 0$

Central height $c = 50$ so $a = -50$

Period $\frac{360}{b} = 20$ so $b = 18$

14 Applications and interpretation: Geometry

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 14A

7

$$P = 2r + \frac{\theta}{360} \times 2\pi r$$
$$= 16 + \frac{35}{360} \times 2\pi \times 8 = 20.9 \text{ cm}$$

$$A = \frac{\theta}{360} \times \pi r^2$$
$$= \frac{35}{360} \times \pi(8)^2 = 19.5 \text{ cm}^2$$

8

$$P = 2r + \frac{\theta}{360} \times 2\pi r$$
$$= 12.4 + \frac{140}{360} \times 2\pi \times 6.2 = 27.5 \text{ cm}$$

$$A = \frac{\theta}{360} \times \pi r^2$$
$$= \frac{140}{360} \times \pi(6.2)^2 = 47.0 \text{ cm}^2$$

9

$$l = \frac{\theta}{360} \times 2\pi r$$
$$= \frac{70}{360} \times 2\pi r = 12.3$$

$$r = 12.3 \times \frac{360}{70 \times 2\pi} = 10.1 \text{ cm}$$

10 a

$$l = \frac{\theta}{360} \times 2\pi r$$
$$= \frac{\theta}{360} \times 10\pi = 7$$

$$\theta = 7 \times \frac{360}{10\pi} = 80.2°$$

b

$$A = \frac{\theta}{360} \times \pi r^2$$
$$= \frac{80.2}{360} \times \pi(5)^2 = 17.5 \text{ cm}^2$$

11

$$A = \frac{\theta}{360} \times \pi r^2$$
$$= \frac{\theta}{360} \times \pi(23)^2 = 185 \text{ cm}^2$$
$$\theta = \frac{185 \times 360}{\pi \times 23^2} = 40.1°$$

12

$$A = \frac{\theta}{360} \times \pi r^2$$
$$= \frac{155}{360} \times \pi r^2 = 326 \text{ cm}^2$$
$$r = \sqrt{\frac{326 \times 360}{\pi \times 155}} = 15.5 \text{ cm}$$

13 a

$$A = \frac{\theta}{360} \times \pi r^2$$
$$= \frac{74.5}{360} \times \pi r^2 = 87.3 \text{ cm}^2$$
$$r = \sqrt{\frac{87.3 \times 360}{74.5 \times \pi}} = 11.6 \text{ cm}$$

b

$$P = 2r + \frac{\theta}{360} \times 2\pi r$$
$$= 23.2 + \frac{74.5}{360} \times 2\pi \times 11.6 = 38.2 \text{ cm}$$

14 For the rectangle part:

$$A = 5 \times 7 = 35 \text{ cm}^2$$
$$P = 7 + 2(5) = 17 \text{ cm}$$

For the sector part:

$$A = \frac{\theta}{360} \times \pi r^2$$
$$= \frac{51.6}{360} \times \pi(7)^2 = 22.1 \text{ cm}^2$$

$$P = r + \frac{\theta}{360} \times 2\pi r$$

$$= 7 + \frac{51.6}{360} \times 2\pi \times 7 = 13.3 \text{ cm}$$

For the whole figure:

$$A = 35 + 22.1 = 57.1 \text{ cm}^2$$

$$P = 17 + 13.3 = 30.3 \text{ cm}$$

15

$$P = 2r + \frac{\theta}{360} \times 2\pi r$$
$$= 2r + \frac{103}{360} \times 2\pi r = 26$$

$$r = \frac{13}{1 + \frac{103\pi}{360}} = 6.85 \text{ cm}$$

16 $A = \frac{\theta}{360} \times \pi r^2 = 18$

$$\theta = \frac{18 \times 360}{\pi r^2} \qquad (1)$$
$$P = 2r + \frac{\theta}{360} \times 2\pi r = 30$$

Substituting (1):

$$2r + \frac{36}{r} = 30$$

$$r^2 - 15r + 18 = 0$$

Using calculator solver: $r = 13.7$ cm or 1.32 cm

17

$$\text{Arc } PQ = \frac{\theta}{360} \times 2\pi r$$
$$= \frac{50}{360} \times 2\pi \times 8 = 6.98 \text{ cm}$$

$$\text{Chord } PQ = \sqrt{8^2 + 8^2 - 2 \times 8^2 \times \cos(50°)} = 6.76 \text{ cm}$$

The difference is $6.98 - 6.76 = 0.219$ cm

18 Shaded region:

$$A = \frac{\theta}{360} \times \pi r^2 - \frac{1}{2} r^2 \sin\theta$$
$$= \frac{40}{360} \times \pi \times 12^2 - \frac{12^2}{2} \sin 40°$$
$$= 3.98 \text{ cm}^2$$

$$P = \frac{\theta}{360} \times 2\pi r + \sqrt{2r^2 - 2r^2\cos\theta}$$
$$= \frac{40}{360} \times 24\pi + \sqrt{2(12^2)(1 - \cos(40°))}$$
$$= 8.38 + 8.20 = 16.6 \text{ cm}$$

19 Inner sector:

$$A = \frac{\theta}{360} \times \pi r^2$$
$$= \frac{60}{360} \times \pi \times 15^2 = 37.5\pi \text{ cm}^2$$

If the shaded region is 16.5π cm^2 then the larger sector has total area 54π cm^2

$$54\pi = \frac{60}{360} \times \pi(15 + x)^2$$

$$(15 + x)^2 = 324$$

$$15 + x = 18$$

$$x = 3$$

20 Slant length of cone is $\sqrt{8^2 + 22^2} = 23.4 = r$

Arc length of sector $\frac{2\pi r\theta}{360} = 2\pi(8)$

Then $\theta = \frac{16\pi(360)}{2\pi r} = 123$

21 If the two points of circle intersections are P and Q then each of triangles C_1PC_2 and C_1QC_2 are equilateral, so the angle subtended by PQ in each case is $120°$

Shaded area is the sum of two equal segments.

Each segment area A can be calculated as the difference between sector area and the triangle between the end points of the chord PQ and the centre of the circle.

$$A = \frac{\theta}{360} \times \pi r^2 - \frac{1}{2} r^2 \sin\theta$$
$$= \frac{120}{360} \times \pi(8)^2 - \frac{8^2}{2} \sin 120° = 39.3 \text{ cm}^2$$

So the shaded area is $2 \times 39.3 = 78.6$ cm^2

The perimeter of the shaded area is twice one of the arc lengths

$$\text{Perimeter} = 2\left(\frac{\theta}{360} \times 2\pi r\right)$$
$$= 2\left(\frac{120}{360} \times 16\pi\right)$$
$$= 33.5 \text{ cm}$$

Exercise 14B

15 Line connecting $(1, 2)$ and $(3, 1)$ has gradient -0.5 .

The boundary line runs perpendicular to that, so has gradient 2 and passes through the midpoint $(2, 1.5)$.

The boundary line has equation $y - 2x = -2.5$

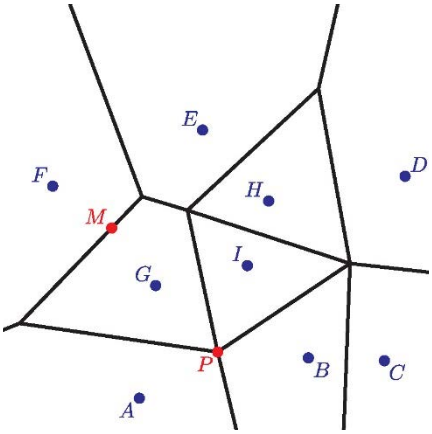
16 **a** $y = 5$

b $(3, 5)$ lies in the cell containing B. Estimate temperature 18°C.

c $(7, 0)$ lies in the region containing G. Estimate air pressure 1025 mbar.

d $(6, 9)$ lies in the region containing E. Estimate rainfall 31 mm.

17



a **i** F or G

ii A, B, G or I

b The point where five regions connect: B, C, D, H and I.

18

coordinates	(1, 2)	(3, 5)	(6, 4)	(0, 9)	(11, 9)	(13, 2)	(9, 9)	(1, 8)
number of plants	3	8	3	11	6	12	2	0
Pond	B	B	D	A	C	C	A	A

Average contamination in each of the pond regions:

$$A: \frac{11+2+0}{3} = 4.3$$

$$B: \frac{3+8}{2} = 5.5$$

$$C: \frac{6+12}{2} = 9$$

$$D: 3$$

It appears that pond C has the highest contamination in its vicinity, so is the likely source of the disease.

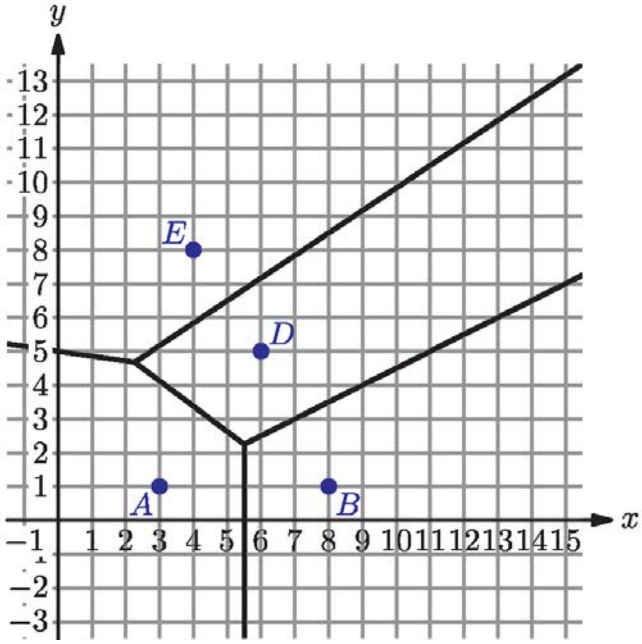
19 **a** **i** D

ii C

iii D

b The cell containing site A contains all the stores served by distribution centre A (all the stores for which A is the closest distribution centre)

c



d (14, 1) will now be served by B.

20 a i Gradient $m_{AB} = 1$ so the perpendicular bisector has gradient -1 through $(2, 2)$.

Equation is $y = 4 - x$

ii Gradient $m_{BC} = 0$ so the perpendicular bisector is vertical through $(3, 4)$.

Equation is $x = 3$

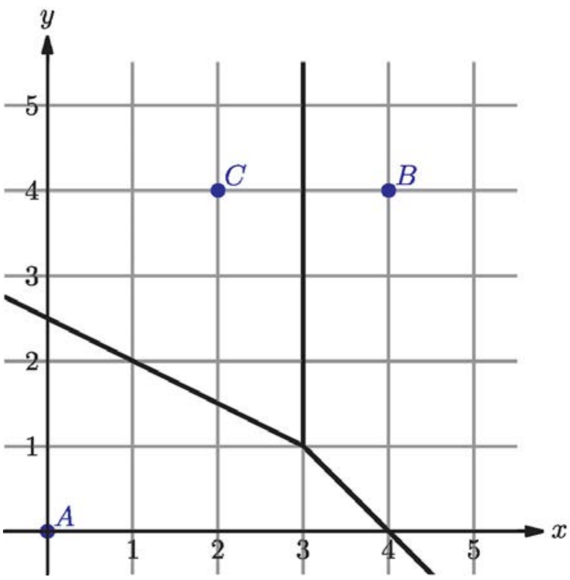
iii Gradient $m_{AC} = 2$ so the perpendicular bisector has gradient $-\frac{1}{2}$ through $(1, 2)$.

Equation is $x + 2y = 5$

b $y = 4 - x$ and $x = 3$ intersect at $(3, 1)$

c This point also lies on the perpendicular bisector of AC, so is the single point equidistant from all three points and is the only vertex on the Voronoi diagram

d



e The point $(4, 2)$ is in the region around B, so is closest to that point.

21 a The gradient of the line connecting B and C has gradient $-\frac{2}{4} = -\frac{1}{2}$

So the perpendicular bisector of that line has gradient 2 and passes through their midpoint $(9, 8)$.

The line equation is $(y - 8) = 2(x - 9)$

Simplifying: $y = 2x - 10$

b This will be the intersection of the three edges on the Voronoi diagram; the intersection of

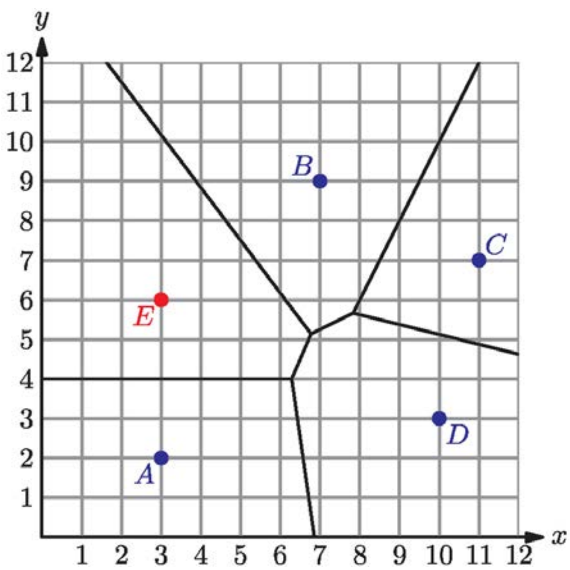
$y = -\frac{1}{4}x + \frac{61}{8}$ and $y = 2x - 10$

$$2x - 10 = -\frac{1}{4}x + \frac{61}{8}$$
$$2.25x = 17.625$$
$$x = 7.83, y = 5.67$$

The point is at $(7.83, 5.67)$

c The points furthest north in the region containing D for integer coordinates are $(7, 5)$, $(8, 5)$and $(9, 5)$; answer **ii** is valid.

d



e The nearest neighbour to $(7, 7)$ is B so estimate the altitude at that point as 412 m.

22 a Line connecting $C(13,6)$ and $D(1,14)$ has gradient $\frac{6-14}{13-1} = -\frac{2}{3}$

The perpendicular bisector, given in red, has gradient $\frac{3}{2}$ and passes through the midpoint $(7, 10)$.

The equation of the red line is $3x - 2y = 1$

b $(8, 13)$ is closest to hospital D.

c Within the city, the two points where multiple edges cross are at either end of the red line: $ACD(7, 10)$ and $BCD(9, 13)$.

ACD lies at distance $\sqrt{6^2 + 4^2} = \sqrt{52}$ from point D.

BCD lies at distance $\sqrt{8^2 + 1^2} = \sqrt{65}$ from point D.

So $(9, 13)$ is furthest from all of A, B, C and D.

23 a $(7, 4)$ is closest to D.

b $(3, 6)$

c The area of the trapezoidal region around A is $\frac{3(7+9)}{2} = 24$

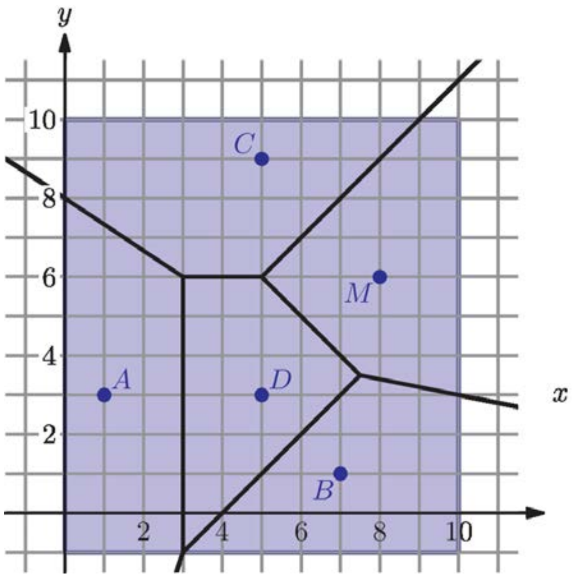
This represents a region with area $2400\ \text{m}^2$

d Line connecting $M(8,6)$ and $D(5,3)$ has gradient $\frac{3}{3} = 1$

So the perpendicular bisector has gradient -1 and passes through the midpoint $(6.5, 4.5)$.

The bisector has equation $x + y = 11$

e



24 a For points $A(0, 0), B(30, 30)$ and $C(50, 10)$.

Perpendicular bisector of A and B has equation $x + y = 30$

Perpendicular bisector of A and C has equation $5x + y = 130$

The intersection of these will be the point of furthest distance from all three towns:

$4x = 100$

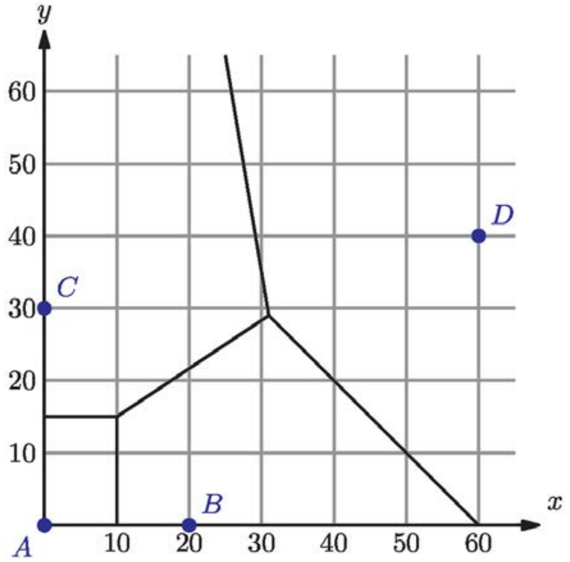
$x = 25, y = 5$

The location for the waste dump is at $(25, 5)$.

b The distance is the same from each city: $\sqrt{25^2 + 5^2} = 5\sqrt{26} = 25.5$ km

c The model assumes that each town can be represented by a single point, and that all space is equally easily travelled, and that any land contours or other features (for example, roads, watercourses) are not relevant to the decision.

25 a $A(0,0), B(20,0), C(0,30), D(60,40)$ produce the following Voronoi diagram:



Using a Voronoi diagram: The vertices of the regions are points of maximal local distance from adjacent towns; the one furthest from all is at the BCD vertex.

Edge BD has equation $y = 60 - x$

Edge BC has equation $y - 15 = \frac{2}{3}(x - 10)$ or $y = \frac{2}{3}x + \frac{50}{6}$

The vertex is the intersection of these edges: $60 - x = \frac{2}{3}x + \frac{50}{6}$

$$\frac{5}{3}x = \frac{310}{6}$$

$$x = 31, y = 29$$

The dump should be located at $(31, 29)$

b The dump would be moved further from town B, so (assuming C and D have similar populations) would move north west along the CD edge.

Mixed Practice

1 a

$$A = \frac{\theta}{360} \times \pi r^2$$
$$= \frac{32}{360} \times \pi(10)^2 = 27.9 \text{ cm}^2$$

b

$$P = 2r + \frac{\theta}{360} \times 2\pi r$$
$$= 20 + \frac{32}{360} \times 2\pi \times 10 = 25.6 \text{ cm}$$

2 If the pizza has been cut into n equal slices then

$$A = \frac{1}{n} \times \pi r^2$$
$$= \frac{\pi(15)^2}{n} = 88 \text{ cm}^2$$
$$n = \frac{\pi(15)^2}{88} = 8.03$$

The pizza has been cut into 8 approximately equal slices.

3

$$V = \frac{3}{4}\pi(0.5)^2 \times 2$$
$$= 1.18 \text{ m}^3$$

4 **a** Gradient of line connecting the capitals is $\frac{53-15}{330-48} = \frac{19}{141}$

So perpendicular has gradient $-\frac{141}{19}$

Midpoint between capitals is $\left(\frac{330+48}{2}, \frac{53+15}{2}\right) = (189, 34)$

Equation of the border is $(y - 34) = -\frac{141}{19}(x - 189)$

$y = -7.42x + 1440$ (to 3 s.f.)

b The closest distance will be from either capital to their midpoint:

$\sqrt{(330 - 189)^2 + (53 - 34)^2} = 142$ km

5 **a** Distance $AB = \sqrt{2^2 + 5^2} = 5.39$ km

b Line connecting $A(6, 6)$ and $B(4, 1)$ has gradient $\frac{6-1}{6-4} = \frac{5}{2} = 2.5$

c The perpendicular bisector of segment AB will have gradient $-\frac{2}{5}$ and pass through the midpoint $(5, 3.5)$.

The equation of this line is $2x + 5y = 27.5$

$4x + 10y = 55$

d P is the point equidistant from feeding stations A, C and D, and so is the point in that region of the diagram furthest from any feeding station.

6 **a** The area of the pentagonal cell containing pump E is 7 km^2

The population density there is therefore $\frac{2215}{7} = 316$ people per km^2.

b Loella lives in the region closest to pump G.

Estimate her probability of getting cholera as $\frac{715}{3689} = 0.194$

c As in part **b**, using the ratio of cases to population as a measure of the probability of cholera:

Pump Label	**Population N**	**Number of cholera cases n**	**Probability $\frac{n}{N}$**
A	4807	318	0.0662
B	1844	624	0.338
C	604	82	0.136
D	1253	125	0.0998
E	2215	322	0.145
F	1803	421	0.233
G	3689	715	0.194

From this, B clearly has the highest density of cholera per head of population and so in the absence of other information is the best location to suggest as the source of the infection.

7 $A(10, 10), B(0, 30), C(20, 40)$

a Line connecting A and B has gradient -2 so the perpendicular bisector has gradient $\frac{1}{2}$ and passes through $(5, 20)$.

This perpendicular bisector has equation $x - 2y = -35$

Line connecting A and C has gradient 3 so the perpendicular bisector has gradient $-\frac{1}{3}$ and passes through $(15, 25)$.

This perpendicular bisector has equation $x + 3y = 90$

The intersection of these two lines is at $(15,25)$, so this would be the point at the common intersection of all the edges of the Voronoi diagram.

The point furthest from all three towns (within the circle of those towns) is $P(15, 25)$.

b distance $AP = \sqrt{5^2 + 15^2} = 15.8$ km

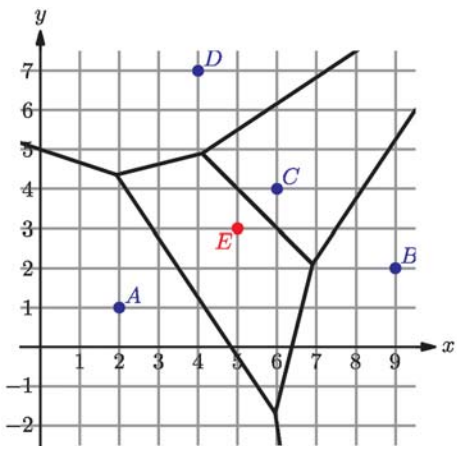
8 **a** Line connecting A and E has gradient $\frac{3-1}{5-2} = \frac{2}{3}$

Then the perpendicular bisector has gradient $-\frac{3}{2}$ and passes through the midpoint $(3.5, 2)$.

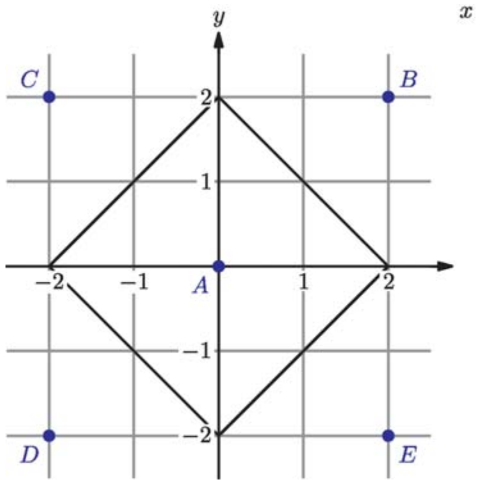
The bisector has equation $3x + 2y = 3(3.5) + 2(2) = 14.5$

$6x + 4y = 29$

b



9



On the Voronoi diagram of the town, the cell containing A is a square of side $\sqrt{8}$ so has area 8 km^2.

Then the number of cellphones which are nearest to A (assuming uniform distribution of the population) is $8 \times 3\,200 = 25\,600$.

10 a i Line connecting A and is vertical so the perpendicular bisector is horizontal and passes through the midpoint $(0, 1)$.

The bisector has equation $y = 1$

ii Line connecting B and C is horizontal so the perpendicular bisector is vertical and passes through the midpoint $(3, 2)$.

The bisector has equation $x = 3$

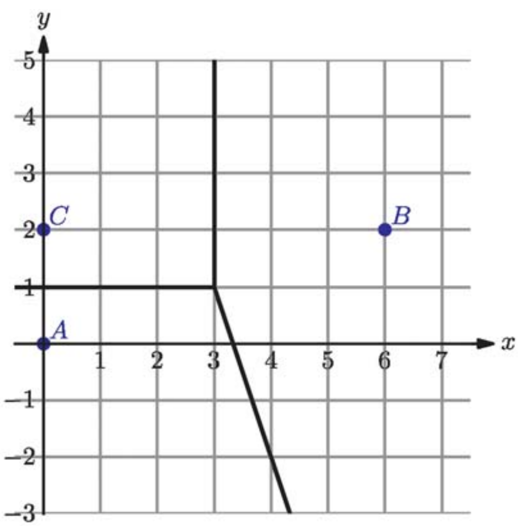
iii Line connecting A and B has gradient $\frac{2}{6} = \frac{1}{3}$

Then the perpendicular bisector has gradient -3 and passes through the midpoint $(3, 1)$.

The bisector has equation $y = 10 - 3x$

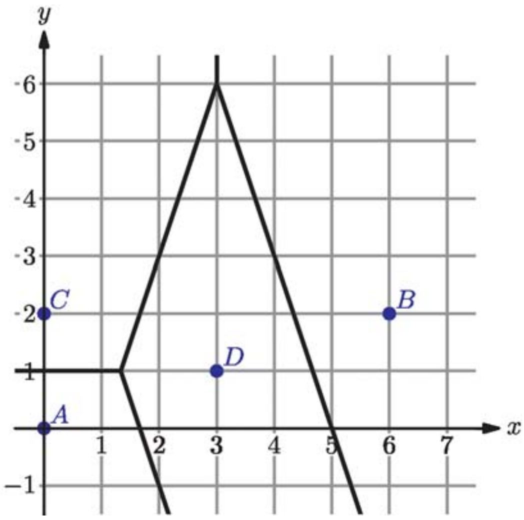
b These three lines meet at $(3, 1)$.

c



d The point furthest from A, B and C within the circle of these points is $(3, 1)$.

e



f Line connecting $C(0,2)$ and $D(3,1)$ has gradient $-\frac{1}{3}$

Then the perpendicular bisector has gradient 3 and passes through the midpoint $(1.5, 1.5)$.

The bisector has equation $3x - y = 3$

This intersects the line $y = 1$ at $\left(\frac{4}{3}, 1\right)$ and the line $x = 3$ at $(3, 6)$.

The area list from store C is the triangle formed by these two points and point D.

The area is $\frac{1}{2} \times \frac{5}{3} \times 5 = \frac{25}{6}$

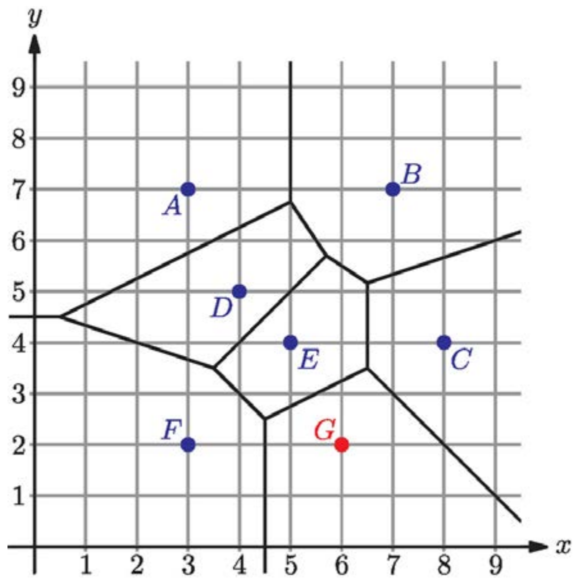
As a percentage of 25, this is 16.7%

11 a i $(2, 6)$ lies in the cell containing A. Estimate temperature as 23 °C.

ii $(8, 3)$ lies in the cell containing C. Estimate temperature as 28 °C.

iii $(6, 3)$ lies in the cell containing E. Estimate temperature as 25 °C.

b



c Point $(6, 3)$ now lies in the region of G so would have estimated temperature 24 °C.

12

$$A = \frac{\theta}{360} \times \pi r^2 = 3\pi$$
$$\theta r^2 = 1080 \qquad (1)$$
$$l = \frac{\theta}{360} \times 2\pi r = \pi$$
$$\theta r = 180 \qquad (2)$$

$(1) \div (2): r = 6$

The radius is 6 cm

13 If the two endpoints of the circle arc are PQ and it is assumed that the centre O of the circle from which the sector is taken lies at the midpoint of the right side of the rectangle then the height of the rectangle is the length of chord PQ and the width of the rectangle equals the radius of the circle.

$$\text{Sector area} = \frac{\pi r^2 \theta}{360} = 0.500r^2 = 7$$
$$r = \sqrt{14} = 3.74 \text{ cm} \approx 37 \text{ mm}$$

Cosine Rule on triangle OPQ

$$PQ = \sqrt{OP^2 + OQ^2 - 2(OP)(OQ)\cos P\hat{O}Q}$$
$$= \sqrt{r^2 + r^2 - 2r^2 \cos 57.3°}$$
$$= 3.59 \text{ cm} \approx 36 \text{ mm}$$

Challenge to student: Find the height and width of the rectangle with least area which can contain this sector, if the sector symmetry line is not held parallel to a rectangle side.

14 a $OB = OC = 4, BC = 3$

Using cosine rule:

$$B\hat{O}C = \cos^{-1}\left(\frac{4^2 + 4^2 - 3^2}{2(4)(4)}\right) = 44.0°$$

b The two shaded areas are equal segments of the larger circle, with subtended angle 44.0°

$$\text{Area} = 2\left(\frac{\theta}{360} \times \pi r^2 - \frac{1}{2} r^2 \sin\theta\right)$$
$$= 2\left(\frac{44.0}{360} \times \pi \times 4^2 - 8\sin 44.0°\right)$$
$$= 1.18 \text{ cm}^2$$

15 a

$$A = \frac{\theta}{360} \times \pi r^2 - \frac{1}{2} r^2 \sin\theta$$
$$= \frac{\theta}{360} 144\pi - 72\sin\theta$$
$$= 0.4\pi\theta - 72\sin\theta = 12$$

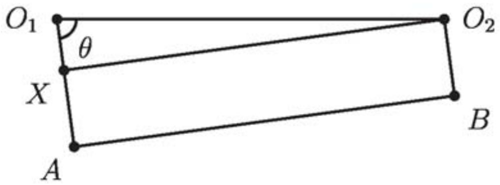
b Using calculator solver: $\theta = 58.3°$

c

$$P = \frac{\theta}{360} \times 2\pi r + \sqrt{2r^2 - 2r^2 \cos\theta}$$
$$= \frac{58.3}{360} \times 24\pi + \sqrt{288 - 288\cos 58.3°} = 23.9 \text{ cm}$$

16 a AB must be perpendicular to both O_1A and O_2B, since the line segment AB is tangent to the circles.

Then O_1O_2BA is a trapezium, with parallel side lengths 6 and 10, perpendicular base and oblique side length 30.



Considering the trapezium as a right-angled triangle O_1XO_2 atop a rectangle O_2XAB, the side lengths of the right triangle are $O_1X = 4, O_1O_2 = 30$

So angle $X\hat{O}_1O_2 = \cos^{-1}\left(\frac{4}{30}\right) = 82.3°$

b Considering the same triangle, $O_2X = \sqrt{30^2 - 4^2} = \sqrt{884} = AB = CD$

Then the bicycle chain consists of the large arc AD, the small arc BC and twice the length of the line segment AB

Since ABO_2O_1 is a trapezium with $O_1A \parallel O_2B$, $B\hat{O}_2O_1 = 180 - A\hat{O}_1O_2 = 97.7°$

$$\text{arc } AD = \frac{360 - 2 \times 82.3}{360} \times 2\pi \times 10 = 34.1 \text{ cm}$$

$$\text{arc } BC = \frac{360 - 2 \times 97.7}{360} \times 2\pi \times 6 = 17.2 \text{ cm}$$

Total bike chain length $= 34.1 + 17.2 + 2\sqrt{884} = 111$ cm

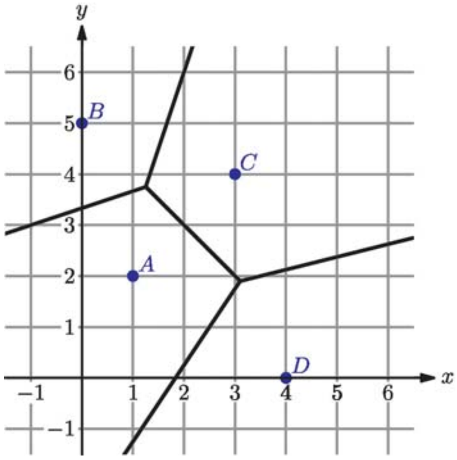
17 a Since DM is the perpendicular bisector of AB and EM is the perpendicular bisector of AC, it follows that angles $A\hat{D}M$ and $A\hat{E}M$ are both 90°

Then $D\hat{M}E = 180 - B\hat{A}C = 130°$

b $\frac{130}{360} \times 100\% = 36.1\%$

c The nearest neighbour to the boat is cabin B so estimate the temperature of the water near the boat as 18.2 °C

18 Using a Voronoi diagram for the four fixed magnets, there are two points where edges meet.



Equations of perpendicular bisectors (edges of the diagram):

$AC: x + y = 5$

$AD: 3x - 2y = 5.5$

$BC: 3x - y = 0$

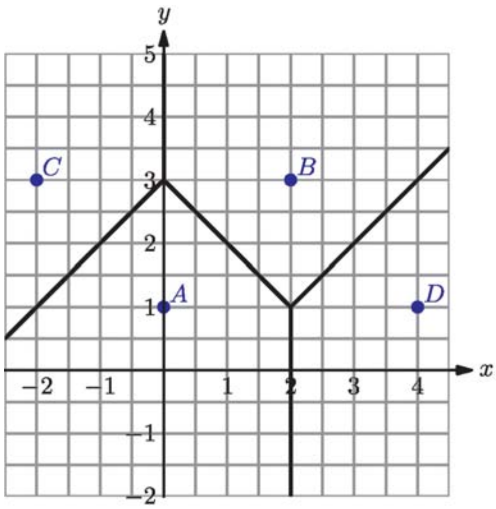
Then using calculator solver, the two intersection points are

$ACD: (3.1, 1.9)$ distance from A: $\sqrt{2.1^2 + 0.1^2} = 2.10$

$ABC: (1.25, 3.75)$ distance from A: $\sqrt{0.25^2 + 1.75^2} = 1.77$

The point equidistant from A, C and D is further from the magnets than the point equidistant from A, B and C (though this would also be locally stable, since any small movement from it would bring the fifth magnet closer to one of the others).

19 a



b Require his path to be a horizontal line segment such that each region has the same length segment within it.

All the oblique edges lie at 45° to the vertical.

For the distance in region A to match that in region B, his path mush be along the line $y = 2$

The distance in section A is then seen to be 2 units.

His total run must therefore be 8 units.

Worked solutions

20 a $OR = \sqrt{a^2 + b^2} = OP = 1$

Then $a^2 + b^2 = 1$

b If $OPQR$ is a rhombus then $\overrightarrow{OP} = \overrightarrow{RQ} = \begin{pmatrix}1\\0\end{pmatrix}$

So Q has coordinates $(a+1, b)$

c OQ has gradient $\frac{b}{a+1}$ so its perpendicular bisector has gradient $-\frac{a+1}{b}$ and passes through the midpoint $\left(\frac{a+1}{2}, \frac{b}{2}\right)$.

Its equation is therefore $(a+1)x + by = \frac{(a+1)^2}{2} + \frac{b^2}{2}$

Simplifying and using the result from part **a**:

$$(a+1)x + by = \frac{1}{2}(a^2 + 2a + 1 + b^2) = a + 1$$

Substituting the coordinates of $P(1,0)$:

$(a+1)(1) + b(0) = a + 1$ so P lies on this line.

Substituting the coordinates of $R(a, b)$:

$(a+1)a + b(b) = a^2 + a + b^2 = a + 1$ so R lies on this line.

15 Applications and interpretation: Hypothesis testing

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 15A

16 Observed values:

	Veggie burger	Fish fingers	Peperoni pizza
Junior school	18	26	51
Senior school	53	38	47

Expected values:

	Veggie burger	Fish fingers	Peperoni pizza
Junior school	18	26	51
Senior school	53	38	47

Degrees of freedom: $(3-1)\times(2-1)=2$

$\chi^2 = 12.14, p = 0.00231 < 0.05$

There is sufficient evidence that age and food choices are not independent.

17 Observed values:

	Ants	Bees	Flies
Meadow	26	15	21
Forest	32	6	18

Expected values:

	Ants	Bees	Flies
Meadow	30.5	11.0	20.5
Forest	27.5	9.97	18.5

Degrees of freedom: $(3-1)\times(2-1)=2$

$\chi^2 = 4.41, p = 0.110 > 0.1$

There is insufficient evidence that the distribution of type of insect is different in the two locations.

18 a Observed values:

Flower type	**A**	**B**	**C**
Number	14	18	28

Expected values:

Flower type	**A**	**B**	**C**
Number	10	20	30

Degrees of freedom: $(3-1)=2$

b $\chi^2 = 1.93$

c $1.93 < 4.605$

There is insufficient evidence that the underlying ratio of flowers is $1:2:3$

19 a H_0: Each course is equally likely

H_1: The courses are not all equally likely

b

Course	Analysis and approaches HL	Analysis and approaches SL	Applications and interpretation HL	Applications and interpretation SL
Observed Frequencies	18	21	8	33
Expected Frequencies	20	20	20	20

Degrees of freedom: $(4-1)=3$

c $\chi^2 = 15.9\ p = 0.00119 < 0.05$

There is sufficient evidence that each course is not equally likely.

20 Observed values:

	Amsterdam	**Athens**	**Houston**	**Johannesburg**
Car	12	25	48	24
Bus	18	33	12	18
Bicycle	46	12	7	53
Walk	38	8	3	21

Expected values:

	Amsterdam	**Athens**	**Houston**	**Johannesburg**
Car	32.9	22.5	20.2	33.4
Bus	24.4	16.7	15	24.9
Bicycle	35.6	24.3	21.9	36.2
Walk	21.1	14.4	13.0	21.5

Degrees of freedom: $(4-1)\times(4-1)=9$

$\chi^2 = 125.8\ p = 8.55\times 10^{-23} \approx 0 \ll 0.05$

There is sufficient evidence that city and mode of transport are dependent.

21

Outcome	1	2	3	4	5	6
Observed Frequency	26	12	16	28	14	24
Expected Frequency	20	20	20	20	20	20

Degrees of freedom: $(6 - 1) = 5$

$\chi^2 = 11.6\, p \approx 0.0407 > 0.02$

There is insufficient evidence that the die is not fair.

22 a $\mathrm{H_0}$: The data is drawn from a $\mathrm{B}(3, 0.7)$.

$\mathrm{H_0}$: The data is not drawn from a $\mathrm{B}(3, 0.7)$.

b

Number of successful serves out of three	0	1	2	3
Observed Frequency	7	28	95	70
Expected Frequency	5.40	37.8	88.2	68.6

Degrees of freedom: $(4 - 1) = 3$

c $\chi^2 = 3.57$

d $3.57 < 6.25$

There is insufficient evidence that data are not drawn from a binomial $\mathrm{B}(3,0.7)$ distribution.

23 a $\mathrm{B}(6,0.5)$

b

Number of tails	0	1	2	3	4	5	6
Frequency	9	62	120	178	152	67	12
Expected frequency	9.375	56.25	140.625	187.5	140.625	56.25	9.375

c Degrees of freedom: $7 - 1 = 6$

$\chi^2 = 7.82 < 12.6$

There is insufficient evidence that the coins are biased.

24 a $X \sim B(4, 0.5)$

b $\mathrm{H_0}$: The data comes from the distribution $\mathrm{B}(4, 0.5)$.

$\mathrm{H_1}$: The data does not come from this distribution.

c $\nu = (5 - 1) = 4$

d

Number of correct answers	0	1	2	3	4
Observed Frequency	12	13	45	22	8
Expected Frequency	6.25	25	37.5	25	6.25

$\chi_4^2 = 13.4, p = 0.00948 < 0.02$

There is sufficient evidence that the data do not come from a $\text{B}(4,0.5)$ distribution.

25 a

Distance (m)	< 5	5 to 6	6 to 7	> 7
Probability	0.159	0.440	0.334	0.0668

b Degrees of freedom $= (4 - 1) = 3$

c H_0: The data is drawn from a normal $\text{N}(5.8, 0.8^2)$ distribution.

H_1: The data is not drawn from a normal $\text{N}(5.8, 0.8^2)$ distribution.

d $\chi^2 = 2.82,\ p = 0.420 > 0.1$

There is insufficient evidence that the data do not come from a $\text{N}(5.8, 0.8^2)$ distribution.

26 a

Time (mins)	< 21.5	$21.5 - 22.5$	$22.5 - 23.5$	$23.5 - 24.5$	> 24.5
Observed Frequency	3	8	14	17	8
Expected Frequency	14.1	7.09	7.62	7.09	14.1

b Degrees of freedom$= (5 - 1) = 4$

c $\chi^2 = 30.7$

d $30.7 > 9.49$

There is sufficient evidence that the data do not come from a $\text{N}(23, 2.6^2)$ distribution.

27

Grade	3	4	5	6	7
Observed Frequency	2	10	9	12	7
Expected Frequency	6.86	8.00	8.57	8.57	8.00

Degrees of freedom$= (5 - 1) = 4$

$\chi^2 = 5.46 \quad p = 0.243 > 0.1$

There is insufficient evidence that the model is not appropriate.

28 a Observed values:

	Vegetarian	Vegan	Eats meat
11–13	12	32	21
14–15	22	16	30
16–19	26	18	23

Expected values:

	Vegetarian	Vegan	Eats meat
11–13	19.5	21.45	24.05
14–15	20.4	22.44	25.16
16–19	20.1	22.11	24.79

H_0: Age and dietary choice are independent among the students at her school.

H_1: Age and dietary choice are not independent among the students at her school.

Degrees of freedom= $(3-1)\times(3-1)=4$

$\chi^2 = 14.0 \quad p = 0.00733 < 0.01$

There is sufficient evidence that diet and age are dependent.

b The values in the first two rows (both expected and observed) remain the same.

Observed values:

	Vegetarian	Vegan	Eats meat
11–13	12	32	21
14–15	22	16	30
16–17	13	12	10
18–19	13	6	13

Expected values:

	Vegetarian	Vegan	Eats meat
11–13	19.5	21.45	24.05
14–15	20.4	22.44	25.16
16–17	20.1	22.11	24.79
18–19			

H_0: Age and dietary choice are independent among the students at her school.

H_1: Age and dietary choice are not independent among the students at her school.

Degrees of freedom= $(4-1)\times(3-1)=6$

$\chi^2 = 15.9 \quad p = 0.0141 > 0.01$

There is insufficient evidence that diet and age are dependent.

29 a Observed values:

	Boys	Girls
Biology	12	16
Chemistry	8	12
Physics	20	12

Worked solutions

Expected values:

	Boys	Girls
Biology	14	14
Chemistry	10	10
Physics	16	16

H_0: Favourite science is independent of gender

H_1: Favourite science is not independent of gender

Degrees of freedom$= (3-1) \times (2-1) = 2$

$\chi^2 = 3.37 \quad p = 0.185 > 0.1$

There is insufficient evidence that favourite science depends on gender

b All values in the tables (observed and expected) are multiplied by 3 for a sample 3 times the size and with exactly the same proportions.

Then the sum of $\frac{(E_i - O_i)^2}{E_i}$ will also be 3 times as large.

$\chi^2 = 10.1 \quad p = 0.00636 \ll 0.1$

There is now sufficient evidence that favourite science depends on gender.

Exercise 15B

10 H_0: $\mu = 34$

H_1: $\mu > 34$

where μ is the true mean amount of time (in minutes) needed to complete homework for the new teacher.

$\bar{x} = 35.5, \qquad \sigma = 7.33$

$t = 0.579, \qquad p = 0.290 > 0.1$

There is insufficient evidence, at the 10% significance level, that the mean amount of time to complete homework is greater than 34 minutes.

11 H_0: $\mu = 300$

H_1: $\mu \neq 300$

where μ is the true mean amount of water in the bottles (in ml).

$\bar{x} = 298, \qquad \sigma = 6$

$t = -1.49, \qquad p = 0.152 > 0.05$

There is insufficient evidence, at the 5% significance level, that the mean amount of water in bottles is different from 300 ml.

12 $H_0: \mu = 23.6$

$H_1: \mu < 23.6$

where μ is the true mean height of trees in the second forest (in m).

$\bar{x} = 21.3, \quad \sigma = 5.92$

$t = -1.23, \quad p = 0.125 > 0.1$

There is insufficient evidence, at the 10% significance level, that the mean height of trees in this second forest is less than 23.6 m.

13 $H_0: \mu = 14.3$

$H_1: \mu > 14.3$

where μ is the true mean temperature in the town (in °C)

$\bar{x} = 17.3, \quad \sigma = 8.6$

$t = 1.91, \quad p = 0.0330 < 0.05$

There is sufficient evidence, at the 5% significance level, that the true mean temperature is greater than 14.3 °C.

14 a $H_0: \mu = 42$

$H_1: \mu \neq 42$

where μ is the true mean time (in minutes) Hamid can now run 10 km.

$\bar{x} = 42.25, \quad \sigma = 2.25$

$t = 0.314, \quad p = 0.763 \gg 0.1$

There is insufficient evidence, at the 10% significance level, that his running time is different from 42 minutes.

b To use a t-test it is assumed that runs times are distributed normally.

15 a Assume that the volumes dispensed are normally distributed.

b $H_0: \mu = 25$

$H_1: \mu \neq 25$

where μ is the true mean volume (in ml) of medicine being dispensed

$\bar{x} = 23.8, \quad \sigma = \sqrt{3.8}$

$t = -3.37, \quad p = 0.00213 < 0.05$

There is sufficient evidence, at the 5% significance level, that the true mean volume dispensed is not 25 ml.

16 Two-sample t-test.

$H_0: \mu_A = \mu_B$

$H_1: \mu_A \neq \mu_B$

where μ_A is the population mean time for breed A and μ_B is the population mean time for breed B.

From GDC: $t = 0.673, \ p = 0.514 > 0.05$

Do not reject H_0; there is insufficient evidence that the two breeds have different mean masses.

Worked solutions

17 a The populations from which the samples are drawn are both normal

To use a pooled t-test, also assume that both populations have the same variance.

b $H_0: \mu_1 = \mu_2$

$H_1: \mu_1 < \mu_2$

where μ_1 is the population mean time for Group 1 and μ_2 is the population mean time for Group 2.

c From GDC: $t = -2.20,\ p = 0.0162$

i $0.0162 < 0.05$

Reject H_0 at 5% significance; there is sufficient evidence that the population from which Group 1 was drawn has a lower mean time than the Group 2 population.

ii $0.0162 > 0.01$

Do not reject H_0 at 1% significance; there is insufficient evidence that the population from which Group 1 was drawn has a lower mean time than the Group 2 population.

18 a $H_0: \mu = 0$

$H_1: \mu > 0$

where μ is the long-term mean daily change in share price.

b Assume that the daily changes are normally distributed.

c From GDC: $t = 0.582,\ p = 0.288 > 0.03$

Do not reject H_0; there is insufficient evidence at the 3% significance level that the mean daily share price change is greater than zero.

19 a To use a pooled t-test, he also must assume that the variances of the two populations are equal.

b $H_0: \mu_1 = \mu_2$

$H_1: \mu_1 < \mu_2$

where μ_1 is the mean sugar in MoccaLite and μ_2 is the mean sugar in ChoccyFroth.

From GDC: $t = -0.9,\ p = 0.197 > 0.1$

Do not reject H_0; there is insufficient evidence at the 10% significance level that the mean amount of sugar in MoccaLite is lower than the amount in ChoccyFroth.

20 a Assumptions: Both populations are normally distributed, with equal variance.

$H_0: \mu_1 = \mu_2$

$H_1: \mu_1 > \mu_2$

where μ_1 is the population mean blood pressure before the trial and μ_2 is the population mean blood pressure after the trial.

$\bar{x}_1 = 146.9, \sigma_1 = 6.76, n_1 = 10$

$\bar{x}_2 = 141.2, \sigma_2 = 6.63, n_2 = 10$

$t = 1.90, \qquad p = 0.0365 < 0.05$

There is sufficient evidence to reject H_0 at 5% significance.

b

	A	B	C	D	E	F	G	H	I	J
Change	0	−6	−8	4	−12	−11	−11	−15	0	2

c If d is the difference (after − before) between measurements for an individual, and μ is the population mean difference then the new hypotheses would be

H_0: $\mu = 0$

H_1: $\mu < 0$

d $\bar{d} = -5.7, \sigma_d = 6.72, n = 10$

$t = 2.68, \qquad p = 0.0125 < 0.05$

There is sufficient evidence to reject H_0 at 5% significance.

e Assumption: The (after − before) differences are normally distributed.

(This assumption follows immediately from the assumption in part **c**, but may be true even if the assumption in part **c** is not true).

f The two sample test requires that the two samples are independent; since this is a before/after measurement pair, this is absolutely not the case. The second test, called a "paired sample t test" is both more appropriate and also more powerful. The second test also has a less restricted assumption, since the differences may be normal even if the separate data lists are not.

Exercise 15C

9 a All points fall on line with positive gradient: B

b Points are strictly ascending but not on a line: C

c Points follow a positive trend but are not on a line or exactly ordered: A

10 a i $r = 0.879$

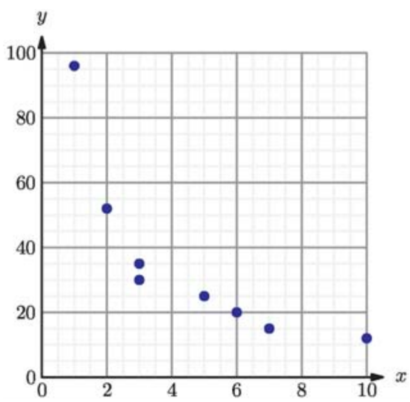
ii $r_s = 0.886$

b There is strong positive correlation.

11 a $r_s = 0.929$

b As temperature rises, the number of people visiting the park also tends to rise (within the data range of temperatures sampled), though not necessarily linearly.

12 a



b The trend is non-linear but does appear reasonably consistently to be a decreasing pattern. Spearman's correlation will capture the pattern of y decreasing as x increases.

c $r_s = -0.994$, showing a strong negative association

13 H_0: There is no correlation

H_1: There is positive correlation

$r_s = 0.303 < 0.564$ so do not reject H_0. There is insufficient evidence of positive correlation.

14 $r_s = 0.332$ which indicates a weak positive association. There is no significant evidence of correlation between time spent playing video games and sleep.

15 a $r_s = -0.952$

b The data supports a statement that older pupils *usually* run faster, but if his statement were exactly true then with all ages different, the value would be $r_s = -1$.

16 a $r_s = 0.8857$

b H_0: There is no correlation between the two sets of marks

H_1: There is correlation between the two sets of marks (two tailed test)

$0.8857 < 0.886$ so do not reject H_0 at the 5% significance level; there is insufficient evidence of correlation.

17 $r_s = -0.505 > -0.571$

There is insufficient evidence of negative correlation between the number of students taking the two subjects.

18 a Rank data:

Student	Alison	Bart	Chad	Dev	Ejam
Mr Wu	4.5	4.5	1	2.5	2.5
Miss Stevens	5	3.5	1.5	1.5	3.5

$r_s = 0.806$

b H_0: There is no correlation between the two sets of grades.

H_1: There is correlation between the two sets of grades (two-tailed).

$0.806 < 0.9$ so do not reject H_0; there is insufficient evidence of correlation between the two sets of grades.

c H_0: There is no correlation between the two sets of grades.

H_1: There is positive correlation between the two sets of grades (one-tailed).

$0.806 > 0.7$ so do reject H_0; there is sufficient evidence of positive correlation (ie agreement) between the two sets of grades.

19 Rank data:

Rank	1	2	3	4	5	6	7	8
Judge 1	1	6	5	8	2	3	4	7
Judge 2	6	8	1	3	5	4	2	7

$r_s = 0.7381$

b H_0: There is no correlation between the two sets of grades.

H_1: There is correlation between the two sets of grades (two-tailed).

$0.7381 < 0.5240$ so do not reject H_0; there is insufficient evidence of correlation between the two sets of grades.

c H_0: There is no correlation between the two sets of grades.

H_1: There is positive correlation between the two sets of grades (one-tailed).

$0.806 > 0.7$ so do reject H_0; there is sufficient evidence of positive correlation (ie agreement) between the two sets of grades.

20 a The new scatter accords more closely to a positive gradient line, so r will be increased.

b The red point remains the greatest x and greatest y values, so none of the rank data has changed; r_s will be unchanged.

c A linear transformation of data has no effect on correlation.; r_s will be unchanged.

Mixed Practice

1 a $r = 0.946$

b $r_s = 1$

c As c increases, V also increases, with the data lying close to, but not exactly along, a straight line.

2 a $X \sim \mathrm{N}(75, 12^2)$

$\mathrm{P}(X < 57) = 0.0668$

b One-sample t-test

H_0: $\mu = 75$

H_1: $\mu > 75$ (one-tailed)

where μ is the true mean score Juan can achieve in a current affairs quiz.

$t = 3.88, \quad p = 0.00186 < 0.05$

Reject H_0 in favour of H_1; there is sufficient evidence that his mean score in current affairs is greater than 75.

3

Score	1	2	3	4	5	6
Observed frequency	45	57	51	56	47	44
Expected frequency	50	50	50	50	50	50

χ^2 test. Degrees of freedom $= (6 - 1) = 5$

H_0: Every result has equal probability

H_1: Not all sides have the same probability

$\chi^2 = 3.12, \quad p = 0.681$

So, there is insufficient evidence to reject H_0.

4 **a** H_0: The die is unbiased (all outcomes have equal probability).

H_1: The die is biased (not all outcomes have equal probability).

b

Outcome	**1**	**2**	**3**	**4**	**5**	**6**
Observed Frequency	42	38	55	61	46	58
Expected Frequency	50	50	50	50	50	50

c Degrees of freedom: $(6 - 1) = 5$

d $\chi^2 = 8.68,\ p = 0.123 > 0.1$

Fail to reject H_0; there is insufficient evidence that the die is biased.

5 **a** $r_s = 0.857$

b H_0: There is no correlation between hours of sunshine and average daily temperature.

H_1: There is a positive correlation between hours of sunshine and average daily temperature.

c $r_s > 0.571$; there is sufficient evidence of positive correlation.

6 H_0: $\mu = 17.5$

H_1: $\mu > 17.5$ (one-tailed test)

where μ is the true mean journey time.

From GDC:

$t = 1.53, \qquad p = 0.0852 < 0.1$

Reject H_0; there is sufficient evidence that the mean time is greater than 17.5 minutes.

7 **a** H_0: $\mu = 16$

H_1: $\mu < 16$

Where μ is the true mean yield of the apple trees in bushels.

b

$$t = \frac{14.7 - 16}{2.3/\sqrt{10}} = -1.787$$

$P(t_9 < -1.787) = 0.0538 > 0.05$

Insufficient evidence of decrease in yield.

8 **a** H_0: Grade distributions are independent of school.

H_1: Grade distributions are not independent of school.

b Observed grades:

	3	**4 or 5**	**6 or 7**
School A	11	44	36
School B	16	83	40

Expected grades:

	3	4 or 5	6 or 7
School A	10.7	50.2	30.1
School B	16.3	76.8	45.9

Degrees of freedom: $(2-1) \times (3-1) = 2$

$\chi^2 = 3.24, \quad p = 0.198 > 0.05$

Fail to reject H_0; insufficient evidence that the schools have different grade distributions.

9.

Tip: Students are required to use a paired sample t-test, because the test subjects have before and after times given. In a paired sample test, you look at the differences as a single sample and test whether the mean difference is zero. This is a more powerful test than a simple two sample t-test, since it uses the additional information of the two times for each person being linked; this was shown in Exercise 15B question 20; the working is given below.

Competitor	**A**	**B**	**C**	**D**	**E**
Difference (after – before training)	-2	-5	-5	3	-4

H_0: $\mu = 0$

H_1: $\mu < 0$ (one-tailed test)

where μ is the true difference in race times after training.

From GDC: $t = -1.729, \ p = 0.0794 > 0.01$

Do not reject H_0; there is insufficient evidence at the 1% significance that the race times improved after training.

In the current syllabus, you are no longer expected to use this technique, even when the data values are paired; the required working is given below and illustrates how much less effective the test is when the pairing information is disregarded.

H_0: $\mu_1 = \mu_2$

H_1: $\mu_1 > \mu_2$ (one-tailed test)

where μ_1 is the true mean race time before training and μ_2 is the true mean race time after training.

From GDC: $t = 0.618, \ p = 0.277 > 0.01$

Do not reject H_0; there is insufficient evidence at the 1% significance that the race times improved after training.

10 H_0: The data comes from the given distribution

H_1: The data does not come from this distribution

Outcome	**1**	**2**	**3**	**4**	**5**	**6**
Observed Frequency	25	46	64	82	99	104
Expected Frequency	50	50	50	50	50	50

Degrees of freedom: $(6-1) = 5$

$\chi^2 = 4.61, \quad p = 0.465 > 0.05$

Fail to reject H_0; there is insufficient evidence that the die has a different distribution to the one claimed.

11 a $\bar{x} = 791$

b $H_0: \mu = 800$

$H_1: \mu < 800$ (one-tailed test)

where μ is the true mean mass of a loaf from the baker.

$t = -2.63, \ \ p = 0.0137 < 0.1$

Reject H_0; there is sufficient evidence that the true mean mass of a loaf is less than 800 g.

12 a This is not a random sample; since the eggs are collected in succession, they may not be independent.

b $H_0: \mu_1 = \mu_2$

$H_1: \mu_1 \neq \mu_2$ (two-tailed test)

where μ_1 is the true mean mass of eggs from chicken A and μ_2 is the true mean mass of eggs from chicken B.

c Assume that the masses of the eggs are normally distributed for both, and that the variance of egg masses is the same for both chickens.

d $t = -0.439, \ p = 0.666 > 0.05$

Do not reject H_0; there is insufficient evidence that the chickens lay eggs with different mean mass.

13 a $r_s = -0.511$

b H_0: The two sets of scores have no correlation.

H_1: There is some correlation between the scores in maths and those in geography (two-tailed).

c $|-0.511| < 0.564$ so do not reject H_0; there is insufficient evidence of correlation between the two subjects' scores.

14 a $H_0: \mu_A = \mu_B$

$H_1: \mu_A \neq \mu_B$ (two-tailed test)

where μ_A is the true mean lifetime of type A batteries and μ_B is the true mean lifetime of type B.

From GDC: $t = 1.69, \ \ p = 0.103 > 0.1$

Do not reject H_0; there is insufficient evidence that the two types have different mean lifetimes.

b Assume that battery lifetimes follow a normal distribution and the two types have equal variances.

15 a $X \sim B(5, 0.45)$

where X is the number of times Roy hits the target in 5 shots.

b

Outcome	**0**	**1**	**2**	**3**	**4 or 5**
Observed	12	18	34	22	14
Expected	5.03	20.6	33.7	27.6	13.1

c Degrees of freedom $= (5 - 1) = 4$

d H_0: $X \sim B(5, 0.45)$

H_1: X follows a different distribution.

$\chi^2 = 11.2, \qquad p = 0.0249 < 0.05$

Reject H_0; there is sufficient evidence that Roy's belief is incorrect.

16 a H_0: All the coins are fair ($P(\text{tails}) = 0.5$).

H_1: (Some of) the coins are biased.

b

Number of tails	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Frequency	12	34	151	218	223	126	32	4
Expected frequency	6.25	43.75	131.25	218.75	218.75	131.25	43.75	6.25

c Degrees of freedom = $8 - 1 = 7$

$\chi^2 = 14.70$

$P(\chi^2_7 > 14.70) = 0.0401 > 2\%$

d Insufficient evidence at 2% significance that the coin are not fair.

17 a Let X be the number of "shiny" cards in a pack.

$$\bar{x} = \frac{(0 \times 70) + (1 \times 80) + (2 \times 40) + (4 \times 10)}{200} = 1$$

This is consistent with the manufacture claim that $X \sim B(5, 0.2)$.

b i H_0: $X \sim B(5, 0.2)$

H_1: X follows a different distribution.

ii

Number of shiny cards	**0**	**1**	**2**	**3, 4 or 5**
Observed Frequency	70	80	40	10
Expected Frequency				

Degrees of freedom: $(4 - 1) = 3$

$\chi^2 = 0.588$

iii $p = 0.899 \gg 0.05$

iv Do not reject H_0; there is no significant evidence to suggest that the manufacturers claim is untrue.

18 a Difference data: $-1.4, -0.9, 0.2, 0.3, -0.8, -0.2$

$\bar{d} = -0.467$

$H_0: \mu_d = 0$

$H_1: \mu_d < 0$

Where μ_d is the true mean time difference year to year.

One-tailed test

$t = -1.695, \quad p = 0.0754 > 0.05$

Do not reject H_0; there is insufficient evidence at the 5% significance that the average change each year is negative.

b i $H_0: \rho_s = 0$

$H_1: \rho_s < 0$

ii $r_s = -0.857$

iii $|-0.857| > 0.714$

Reject H_0; there is sufficient evidence that as his age increases, there is a tendency for Jamie's 100 m race time to decrease.

19 a $H_0: \mu = 62$

$H_1: \mu < 62$

Where μ is the true mean score.

$t = -1.19, \quad p = 0.118 > 0.05$

Reject H_0; there is sufficient evidence to show that the mean score is less than 62.

b $H_0: X \sim N(62, 144)$.

$H_1: X$ has a different distribution

Where X is the score of a student.

Score (s)	$\leqslant 45$	$45 < s \leqslant 55$	$55 < s \leqslant 65$	$65 < s \leqslant 75$	$s > 75$
Observed Frequency	11	23	22	18	6
Expected Frequency	6.26	16.1	25.5	21.0	11.1

Degrees of freedom $= (5 - 1) = 4$

$\chi^2 = 9.79, \quad p = 0.0440 < 0.05$

Reject H_0; there is sufficient evidence that the data are not drawn from $N(62, 144)$.

c The distribution could still be normal, but with a different mean and/or variance than claimed.

d The scores given are discrete, whereas the normal distribution is continuous.

Tip: Not an issue in this case, but also always be alert for a normal distribution model which predicts a large proportion of results outside the possible values (in this context, if the normal model predicted a noticeable number of students scoring more than 100% or less than 0%, this would be a standard objection).

20 a Assuming the scores follow an approximately normal distribution:

$H_0: \mu = 3.3$

$H_1: \mu \neq 3.3$ (two-tailed test)

Where μ is the true mean score.

$t = 2.51, \quad p = 0.0129 < 0.1$

Reject H_0; there is sufficient evidence to conclude that the mean score has changed from the previous year's average.

b $H_0: \rho_s = 0$

$H_1: \rho_s > 0$ (one-tailed test)

$r_s = 0.696 > 0.657$

Reject H_0; there is sufficient evidence to show that higher scores are more popular.

c H_0: Each score has equal probability.

H_1: It is not the case that each score has equal probability.

Score	**1**	**2**	**3**	**4**	**5**	**6**
Observed Frequency	25	30	32	27	34	32
Expected Frequency	30	30	30	30	30	30

Degrees of freedom $= (6 - 1) = 5$

$\chi^2 = 1.93, \quad p = 0.858$

Do not reject H_0; there is insufficient evidence to show that all scores are not equally likely.

16 Applications and interpretation: Calculus

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 16A

12

$$y = x^4 + bx^2 + c$$
$$\frac{dy}{dx} = 4x^3 + 2bx$$
$$\frac{dy}{dx}(1) = 4 + 2b = 0$$
$$b = -2$$
$$y(1) = 1 + b + c = 2$$
$$c = 3$$

13

$$y = x^4 + bx + c$$
$$\frac{dy}{dx} = 4x^3 + b$$
$$\frac{dy}{dx}(1) = 4 + b = 0$$
$$b = -4$$
$$y(1) = 1 + b + c = -2$$
$$c = 1$$

14 a From calculator, roots of $f(x)$ are $x = -0.329$ or 1.54

b $f'(x) = 4x^3 - 3$

From calculator, roots of $f'(x)$ are $x = 0.909$

15

$$y = 4x^3 - x^4$$
$$\frac{dy}{dx} = 12x^2 - 4x^3 = 4x^2(3 - x)$$

Gradient is zero at $x = 0$ and $x = 3$

Horizontal tangents occur at $(0,0)$ and $(3,27)$

16

$$y = 2x^{-3} - 3x^2$$
$$\frac{dy}{dx} = -6x^{-4} - 6x = -6x^{-4}(1 + x^3)$$

Gradient is zero only at $x = -1$

The local maximum is at $(-1, -5)$

17 **a**

$$f(x) = 2x^3 + 5x^2 - 2x + 8$$
$$f'(x) = 6x^2 + 10x - 2$$
$$f'(x) = 0 \text{ at } x = -1.85 \text{ or } 0.180$$

For a positive cubic, the second stationary point is the local minimum

$f(0.180) = 7.81$

The local minimum is $(0.180, 7.81)$

b Checking end values:

$f(-4) = -32$

$f(1) = 13$

The least value of $f(x)$ over domain $-4 \leqslant x \leqslant 1$ is -32.

18 $f(x) = 3x^2 + x^3 - 0.3x^5$

From calculator, local maximum in $-3 \leqslant x \leqslant 3$ is $(1.75, 7.33)$:

Checking end values:

$f(-3) = 108.9$

$f(3) = -36.9$

The greatest value in the given domain is 108.9

19 $f(x) = 3x^2 + \frac{11}{x}$

From calculator, local minimum is at $(1.22, 13.5)$.

Checking end values:

$f(1) = 14$

$f(3) = 30.7$

The range is $13.5 \leqslant f(x) \leqslant 30.7$

20 $f(x) = 3x^4 - 4x^3 - 12x^2 + 2$

From calculator, local minima are at $(-1, -3)$ and $(2, -30)$.

Local maximum is less than end values for a positive quartic.

The range is $f(x) \geqslant -30$

21

$$y = ax^3 - bx^2 - 4x$$
$$\frac{dy}{dx} = 3ax^2 - 2bx - 4$$

Gradient is zero at $(2, -12)$

$$\frac{dy}{dx}(2) = 12a - 4b - 4 = 0$$
$$3a - b = 1 \quad (1)$$
$$y(2) = 8a - 4b - 8 = -12$$
$$2a - b = -1 \quad (2)$$

$(1) - (2)$: $a = 2$ so $b = 5$

22

$$y = ax^2 - bx^{-1}$$
$$\frac{dy}{dx} = 2ax + bx^{-2}$$

Gradient is zero at $(-1,9)$

$$\frac{dy}{dx}(-1) = -2a + b = 0$$
$$b - 2a = 0 \quad (1)$$
$$y(-1) = a + b = 9$$
$$a + b = 9 \quad (2)$$

$(2) - (1)$: $3a = 9$ so $a = 3$ and $b = 6$

23

$$y = ax^5 + 4x^2 - bx$$
$$\frac{dy}{dx} = 5ax^4 + 8x - b$$

Gradient is zero at $(1, -8)$

$$\frac{dy}{dx}(1) = 5a + 8 - b = 0 \quad (1)$$
$$y(1) = a + 4 - b = -8 \quad (2)$$
$$(1) - (2): 4a + 4 = 8$$

$a = 1$ and $b = 13$

From calculator, local maximum of $y = x^5 + 4x^2 - 13x$ is at $(-1.49, 20.9)$.

Exercise 16B

1

$$P' = 20x - 40x^3$$
$$= 20x(1 - 2x^2)$$

Only local maximum for positive x is at $x = \frac{1}{\sqrt{2}}$

$$P\left(\frac{1}{\sqrt{2}}\right) = 2.5$$

Maximum profit is \$2.5 million.

2 From calculator, the maximum for $P = 20n - 3n^2 - n^5$ for integer $1000n$

Is $P(1.256) = 17.26$

The maximum profit is $1726

3 From calculator, the maximum for

$$F = (3 \times 10^{-6})v^3 - (1.2 \times 10^{-4})v^2 - 0.035v + 12$$

Is $F(77.1) = 9.96$

At 77.1 km h^{-1} the fuel consumption per 100 km is minimised.

4 $R = 6t^{-1} - 47t^{-4}, t \geq 2$

Maximum R is $R(3.15) = 1.43$

Maximum growth rate occurs at time 3.15 hours.

5 **a** $P = 40$ cm

b $A = x(20 - x)$

Negative quadratic has its maximum midway between the roots: $A(10) = 100\ \text{cm}^2$.

6 **a** $A = 42\ \text{cm}^2$

b $P = 6x + 28x^{-1}$

From calculator, minimum for positive x is $P(2.16) = 25.9$ cm.

7 $V = x^2(9 - x)$

From calculator, maximum for positive x is $V(6) = 108\ \text{cm}^3$.

8 **a** Other side has length $36x^{-1}$ cm.

$$P = 2x + \frac{72}{x}$$

b From calculator, minimum P for positive x is $P(6) = 24$ cm.

9 Let x be one of the two parallel sides of fencing

The side opposite the house has length $45 - 2x$

$$A = x(45 - 2x)$$

Negative quadratic has maximum midway between the roots so maximum A is $A(11.25) = 253\ \text{m}^2$.

10 **a** Height of box is x, length is $20 - 2x$ and width is $25 - 2x$

$$\begin{aligned} V &= x(20 - 2x)(25 - 2x) \\ &= x(4x^2 - 90x + 500) \\ &= 4x^3 - 90x^2 + 500x \end{aligned}$$

b From calculator, maximum V for positive x is $V(3.68) = 821\ \text{cm}^3$.

11 a

$$V = hx^2 = 460$$
$$h = \frac{460}{x^2}$$
$$S = 2x^2 + 4xh$$
$$= 2x^2 + \frac{1840}{x^1}$$

b From calculator, minimum S for positive x is $S(9.73) = 284$.

So the value of x for which the cuboid has smallest surface area is 9.73.

12 a

$$V = x(2x)h = 2x^2h = 225$$
$$h = \frac{225}{2x^2}$$
$$S = 2(xh) + 2(2xh) + 2(2x^2)$$
$$= 6xh + 4x^2$$
$$= \frac{675}{x} + 4x^2$$

b From calculator, minimum S for positive x is $S(4.39) = 231\ \text{cm}^2$.

When $x = 4.39$ cm, $2x = 8.77$ cm and $h = 112.5x^{-2} = 5.85$ cm.

The box dimensions are 4.39 cm × 8.77 cm × 5.85 cm.

13 a

$$V = \pi r^2 h = 500$$
$$h = \frac{500}{\pi r^2}$$
$$S = 2\pi r^2 + 2\pi rh$$
$$= 2\pi r^2 + \frac{1000}{r}$$

b From calculator, minimum S for positive r is $S(4.30) = 349\ \text{cm}^2$.

$$r = 4.30, h = \frac{500}{\pi r^2} = 8.60$$

c Modelling the bottle as an exact cylinder does not take into account the shaping of a standard bottle (neck, indented base) or the thickness of the plastic.

14 a

$$S = 2\pi r^2 + 2\pi rh$$
$$= 2\pi r(r + h) = 200\pi$$
$$h = \frac{100}{r} - r$$
$$V = \pi r^2 h$$
$$= 100\pi r - \pi r^3$$

b From calculator, maximum volume for positive r is $V(5.77) = 1210\ \text{cm}^3$.

15

$$\text{Total profit } P = x(23 - 0.5x - 2x^{-2}) = 23x - 0.5x^2 - 2x^{-1}$$

From calculator, maximum P for positive x is $P(23) = 264$

23 hats per week would maximise profit, according to this model.

16 a

$$P = x\left(\left(22 - \frac{x}{3}\right) - \left(9 - \frac{x}{6}\right)\right)$$
$$= 13x - \frac{x^2}{6}$$

b From calculator, maximum P for positive x is $P(39) = 253.5$.

For maximum profit, the model predicts that 39 key rings should be produced.

17 a

$$\text{Profit } P = (1000 - 100x)(x - 4) = 100(10 - x)(x - 4)$$

Roots are $x = 4, 10$

Negative quadratic has maximum midway between roots, so maximum profit occurs at $\$x = \$7$.

b The new model tapers the number of units sold, but always predicts that a non-negative number will be sold. The original has negative sales predicted for $x > 10$, which would be unrealistic.

c Profit $P = \frac{1000(x-4)}{(x+1)^2}$

From calculator, maximum P for positive x is $P(9) = 50$.

The optimal price under the new model is \$9.

18 a

$$\text{Distance } d = \text{Speed} \times \text{time} = \left(4 - \frac{1}{m}\right)\left(\frac{200}{m}\right) = \frac{800}{m} - \frac{200}{m^2}$$

b From calculator, maximum d for positive m is $d(0.5) = 800$

Optimal mass is 0.5 kg.

19

$$S = 2\pi r^2 + 2\pi rh = 2\pi r(r + h) = 3850\pi$$

$$h = \frac{1925}{r} - r$$

$$V = \pi r^2 h = 1925\pi r - \pi r^3$$

b From calculator, maximum volume for positive r is $V(25.3) = 102\,000\ \text{cm}^3$.

20 For cone with radius r and height h, slant length $l = \sqrt{h^2 + r^2}$

$$S = \pi r l = 23$$

$$l = \frac{23}{\pi r}$$

$$h = \sqrt{l^2 - r^2}$$

$$= \sqrt{\left(\frac{23}{\pi r}\right)^2 - r^2}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{r}{3}\sqrt{529 - \pi^2 r^4}$$

From calculator, maximum volume for positive r is $V(2.06) = 12.9\text{ cm}^3$.

Exercise 16C

8 Trapezoidal rule approximation:

$x_0 = 4, n = 4, h = 1$

x	y	$\times$	$=$
$x_0 = 4$	0	$\times 1$	0
$x_1 = 5$	1	$\times 2$	2
$x_2 = 6$	1.414	$\times 2$	2.828
$x_3 = 7$	1.732	$\times 2$	3.464
$x_4 = 8$	2	$\times 1$	2
	TOTAL		10.293
	TOTAL $\times \frac{h}{2}$		5.146

Approximate area: 5.15

9 Trapezoidal rule approximation:

$x_0 = 0, n = 5, h = 0.6$

x	y	$\times$	$=$
$x_0 = 0.0$	1	$\times 1$	1
$x_1 = 0.6$	1.483	$\times 2$	2.966
$x_2 = 1.2$	1.844	$\times 2$	3.688
$x_3 = 1.8$	2.145	$\times 2$	4.290
$x_4 = 2.4$	2.408	$\times 2$	4.817
$x_5 = 3.0$	2.646	$\times 1$	2.646
	TOTAL		19.406
	TOTAL $\times \frac{h}{2}$		5.822

Approximate area: 5.82

10 Trapezoidal rule approximation:

$x_0 = 0.5, n = 6, h = 0.5$

x	$f(x)$	$\times$	$=$
$x_0 = 0.5$	4.1	$\times 1$	4.1
$x_1 = 1.0$	5.4	$\times 2$	10.8
$x_2 = 1.5$	5.0	$\times 2$	10.0
$x_3 = 2.0$	4.6	$\times 2$	9.2
$x_4 = 2.5$	4.1	$\times 2$	8.2
$x_5 = 3.0$	3.6	$\times 2$	7.2
$x_6 = 3.5$	3.4	$\times 1$	3.4
	TOTAL		52.9
	TOTAL $\times \frac{h}{2}$		13.225

Approximate area: 13.2

11 a Trapezoidal rule approximation:

$x_0 = 0.5, n = 4, h = 1.5$

x	y	$\times$	$=$
$x_0 = 0.0$	0.3	$\times 1$	0.3
$x_1 = 1.5$	1.8	$\times 2$	3.6
$x_2 = 3.0$	3.2	$\times 2$	6.4
$x_3 = 4.5$	2.1	$\times 2$	4.2
$x_4 = 6.0$	0.3	$\times 1$	0.3
	TOTAL		14.8
	TOTAL $\times \frac{h}{2}$		11.1

Approximate area: 11.1 m^2

b The real area is larger than this estimate, because the chords of the curve which form the upper sides of the trapezoids will all lie below the curve (the curve of the tunnel is concave-down).

12 Trapezoidal rule approximation:

$x_0 = 0, n = 5, h = 0.6$

x	y	$\times$	$=$
$x_0 = 0.0$	0	$\times 1$	0
$x_1 = 0.6$	0.864	$\times 2$	1.728
$x_2 = 1.2$	2.592	$\times 2$	5.184
$x_3 = 1.8$	3.888	$\times 2$	7.776
$x_4 = 2.4$	3.456	$\times 2$	6.912
$x_5 = 3.0$	0	$\times 1$	0
	TOTAL		21.600
	TOTAL $\times \frac{h}{2}$		6.480

Approximate area: 6.48

$$\int_0^3 (3x^2 - x^3)\,\mathrm{d}x = \left[x^3 - \frac{1}{4}x^4\right]_0^3$$
$$= 27 - \frac{81}{4}$$
$$= 6.75$$

$$\text{Percentage error} = \frac{|\text{true value} - \text{approximate value}|}{\text{true value}} \times 100\%$$
$$= \frac{|6.75 - 6.48|}{6.75}$$
$$= 4\%$$

13 Each strip has the equivalent approximate area to a trapezoid with side lengths as shown.

The edge triangles can be considered as trapezoids with one side length being zero.

Trapezoidal rule approximation:

$x_0 = 0.5, n = 6, h = 0.5$

x	y	$\times$	$=$
$x_0 = 0$	0	$\times 1$	0
$x_1 = 2$	8.6	$\times 2$	17.2
$x_2 = 4$	12.3	$\times 2$	24.6
$x_3 = 6$	21.5	$\times 2$	43
$x_4 = 8$	17.2	$\times 2$	34.4
$x_5 = 10$	9.1	$\times 2$	18.2
$x_6 = 12$	0	$\times 1$	0
	TOTAL		137.4
	TOTAL $\times \frac{h}{2}$		137.4

Approximate area: 137 m^2

14 Each strip has the equivalent approximate area to a trapezoid with side lengths as shown.

Trapezoidal rule approximation:

$x_0 = 32, n = 4, h = 10$

x	y	$\times$	$=$
$x_0 = 32$	62	$\times 1$	62
$x_1 = 42$	76	$\times 2$	152
$x_2 = 52$	52	$\times 2$	104
$x_3 = 62$	64	$\times 2$	128
$x_4 = 72$	64	$\times 1$	64
	TOTAL		510
	TOTAL $\times \frac{h}{2}$		2550

Approximate area: 2550 m^2

15 a Trapezoidal rule approximation:

$x_0 = 2, n = 5, h = 0.4$

x	y	$\times$	$=$
$x_0 = 2.0$	0	$\times 1$	0
$x_1 = 2.4$	0.632	$\times 2$	1.265
$x_2 = 2.8$	0.894	$\times 2$	1.789
$x_3 = 3.2$	1.095	$\times 2$	2.191
$x_4 = 3.6$	1.265	$\times 2$	2.53
$x_5 = 4.0$	1.414	$\times 1$	1.414
	TOTAL		9.189
	TOTAL $\times \frac{h}{2}$		1.838

Approximate area: 1.84

b This is an underestimate, because the curve is concave down.

Mixed Practice

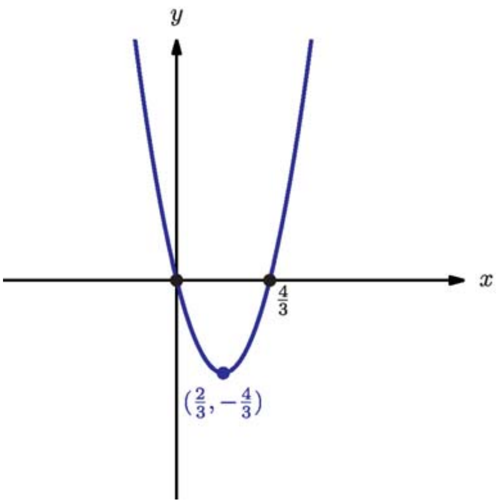
1 a $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x - 4$

Gradient is zero at $x = \frac{2}{3}$

$$y\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) = -\frac{4}{3}$$

$$A: \left(\frac{2}{3}, -\frac{4}{3}\right)$$

b



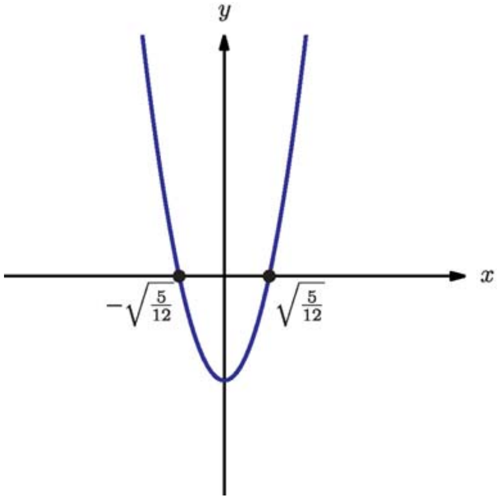
2 $y = 4x^2 - bx$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 8x - b = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(-2) = -16 - b = 0$$

$$b = -16$$

3 **a** $f'(x) = 12x^2 - 5$

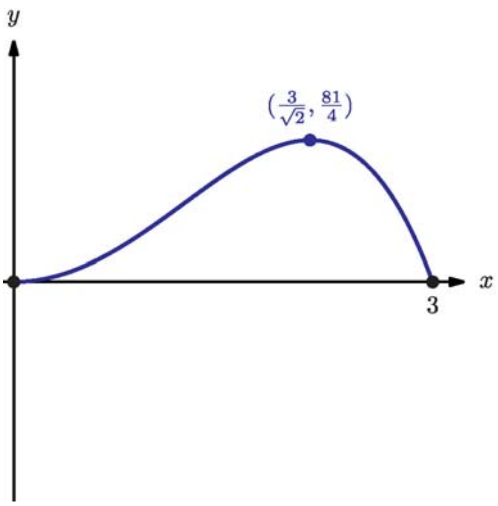


b $x = \pm\sqrt{\frac{5}{12}}$

4 Stationary point on the curve $y = 4x^2 - \frac{5}{x}$ is at $x = -0.855$

5 From calculator, the minimum of $y = 4x^3 - 3x + 8$ is at $(0.5,7)$

6 **a**



Maximum at $\left(\frac{3}{\sqrt{2}}, \frac{81}{4}\right)$, mimima in the interval at the end points $(0, 0)$ and $(3, 0)$.

b Trapezoidal rule approximation:

$x_0 = 0, n = 6, h = 0.5$

x	y	$\times$	$=$
$x_0 = 0.0$	0	$\times 1$	0
$x_1 = 0.5$	2.1875	$\times 2$	4.375
$x_2 = 1.0$	8	$\times 2$	16
$x_3 = 1.5$	15.1875	$\times 2$	30.375
$x_4 = 2.0$	20	$\times 2$	40
$x_5 = 2.5$	17.1875	$\times 2$	34.375
$x_6 = 3.0$	0	$\times 1$	0
	TOTAL		125.125
	TOTAL $\times \frac{h}{2}$		31.28125

Approximate area: 31.3

7 Trapezoidal rule approximation:

$x_0 = 2, n = 5, h = 2$

x	y	$\times$	$=$
$x_0 = 2$	0	$\times 1$	0
$x_1 = 4$	0.693	$\times 2$	1.386
$x_2 = 6$	1.099	$\times 2$	2.197
$x_3 = 8$	1.386	$\times 2$	2.773
$x_4 = 10$	1.609	$\times 2$	3.219
$x_5 12$	1.792	$\times 1$	1.792
	TOTAL		11.367
	TOTAL $\times \frac{h}{2}$		11.367

Approximate area: 11.4

8 **a**

$$2h + 2w = 88$$
$$h = 44 - w$$
$$A = hw = 44w - w^2$$

b Negative quadratic has its maximum midway between the roots.

Maximum A is at $w = 22, h = 22$ (ie when the rectangle is a square).

c $A = 22^2 = 484 \text{ cm}^2$

9 From calculator, minimum L for positive v is $L(65.8) = 17.2$

Optimum speed for fuel consumption is 65.8 km h^{-1}.

10 **a** Edges: $4(2x) + 4(x) + 4(y) = 48$

$$3x + y = 12$$
$$y = 12 - 3x$$

b

$$V = (2x)(x)(y)$$
$$= 2x^2(12 - 3x)$$
$$= 24x^2 - 6x^3$$

c $\frac{dV}{dx} = 48x - 18x^2$

d V is a negative cubic with a double root at $x = 0$, so the maximum for positive x is the non-zero root of $\frac{dV}{dx}$.

V is maximum at $x = \frac{48}{18} = \frac{8}{3} = 2.67$ m

e $V\left(\frac{8}{3}\right) = 56.9\ \text{m}^3$

f Length $2x = 5.33$ m, height $y = 4$ m

g

$$\begin{aligned}\text{Surface area} &= 2(2x \times x) + 2(2x \times y) + 2(x \times y)\\ &= 4x^2 + 6xy\end{aligned}$$

$$x = \frac{8}{3}, y = 4 : S = 92.4\ \text{m}^2$$

$$90 = 15 \times 6 < S < 105 = 15 \times 7$$

To paint the container requires 7 cans of paint.

11

$$\frac{dy}{dx} = 2x + 3x^{-2} = 0$$

$$x^3 = -\frac{3}{2}$$

$$x = \left(-\frac{3}{2}\right)^{\frac{1}{3}} = -1.14$$

The point is $(-1.14, 3.93)$

12

$$\begin{aligned} y &= ax^2 - 48x + b\\ \frac{dy}{dx} &= 2ax - 48\\ \frac{dy}{dx}(8) &= 16a - 48 = 0\\ a &= 3\\ y(8) &= 64a - 384 + b = 21\\ b &= 213\end{aligned}$$

13

$$\begin{aligned} y &= 3x^2 - ax^{-1}\\ \frac{dy}{dx} &= 6x + ax^{-2}\\ \frac{dy}{dx}(-1) &= -6 + a = 0\\ a &= 6\\ y(-1) &= 3 + a = 9\\ \frac{dy}{dx} &= 6x^{-3}(x^3 + 1)\end{aligned}$$

The only stationary point is at $x = -1$, so the local minimum is $(-1, 9)$.

14 a Trapezoidal rule approximation:

$x_0 = 0, n = 4, h = 1$

x	y	$\times$	$=$
$x_0 = 0$	1	$\times 1$	1
$x_1 = 1$	0.368	$\times 2$	0.736
$x_2 = 2$	0.135	$\times 2$	0.271
$x_3 = 3$	0.050	$\times 2$	0.100
$x_4 = 4$	0.018	$\times 1$	0.018
	TOTAL		2.124
	TOTAL $\times \frac{h}{2}$		1.062

Approximate area: 1.062

b From calculator:

$$\int_0^4 \mathrm{e}^{-x}\,\mathrm{d}x = 0.982 \text{ (to 3 s.f.)}$$

$$\text{Percentage error} = \frac{|\text{true value} - \text{approximate value}|}{\text{true value}} \times 100\%$$
$$= \frac{|0.9817 - 1.062|}{0.9817} \times 100\%$$
$$= 8.20\%$$

15 a From calculator: local maximum of the cubic is at $\mathrm{f}(0.423) = 0.385$

The coordinates are $(0.423, 0.385)$.

b End values for the domain are $\mathrm{f}(0) = 0$ and $\mathrm{f}(3) = 6$ so the greatest value for this domain is 6.

16 $0.2x^4 - 3x^3 + 7.5x^2 + 1.3x + 1$

From calculator, the local minimum in the interval is at $(9.19, -256)$.

The end points are $(0,1)$ and $(10, -236)$ so the minimum value in the interval is -256.

17 Cuboid:

$V = x \times x \times y = 300$ so $y = 300x^{-2}$

$S = 2x^2 + 4xy = 2x^2 + 1200x^{-1}$

From calculator, minimum S for positive x is $S(6.69) = 269\text{ cm}^2$.

Cylinder:

$V = \pi r^2 h = 300$ so $h = 300(\pi r^2)^{-1}$

$S = 2\pi r^2 + 2\pi r h = 2(\pi r^2 + 300r^{-1})$

From calculator, minimum S for positive r is $S(3.63) = 248\text{ cm}^2$.

The cylinder offers the smallest possible surface area, with $r = 3.63$ cm and $h = 7.26$ cm.

18 a $V = 20lw$

b

$$20lw = 3000$$
$$l = \frac{3000}{20w} = \frac{150}{w}$$

c

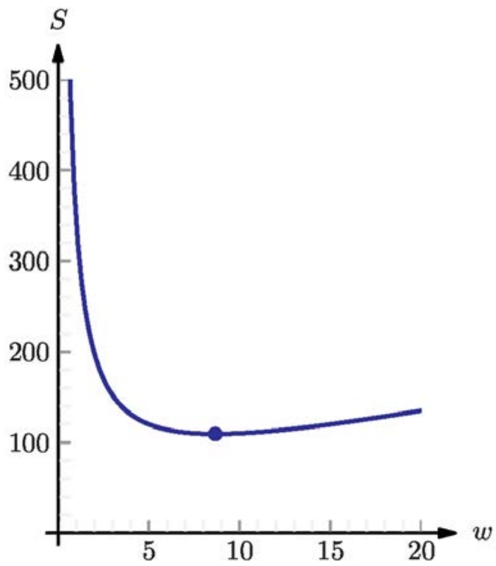
Length of string is $2l + 4w + 2(20) = 2(150w^{-1}) + 4w + 40$
$= 300w^{-1} + 4w + 40$

Tip: There is no indication in the question or the mark scheme to justify why $0 < w \leq 20$.

That being the case, no comment is needed.

In most similar circumstances, you may need to justify the domain.

d



e $\frac{dS}{dw} = 4 - \frac{300}{w^2}$

f S is minimum for positive w at $w = \sqrt{75} = 8.66$

g $l = \frac{150}{w} = 17.3$

h $S(8.66) = 109$ cm

19 a

$$V = \pi x^2 h = 600$$
$$h = \frac{600}{\pi x^2}$$

b i

$$CSA = 2\pi x h$$
$$= \frac{1200}{x}$$

ii $A = 2\pi x^2 + \frac{1200}{x}$

c $\frac{dA}{dx} = 4\pi x - \frac{1200}{x^2}$

d

$$\frac{dA}{dx} = 0: x^3 = \frac{1200}{4\pi}$$
$$x = \sqrt[3]{\frac{300}{\pi}} = 4.57$$

e $A(4.57) = 394 \text{ cm}^2$

20

$$y = ax^3 + 2x^2 + bx + 45$$
$$\frac{dy}{dx} = 3ax^2 + 4x + b$$
$$\frac{dy}{dx}(2) = 12a + 8 + b = 0 \qquad (1)$$
$$y(2) = 8a + 8 + 2b + 45 = 5$$
$$8a + 2b + 48 = 0(2)$$
$$2(1) - 3(2): -4b - 128 = 0$$
$$b = -32$$
$$(1): 12a = -b - 8 = 24 \text{ so } a = 2$$
$$y = 2x^3 + 2x^2 - 32x + 45$$

From calculator, this cubic has local minimum at $(2,5)$ and local maximum at $(-2.67, 107)$

21 a

$$T = at^3 + bt^2 + ct + d$$
$$T(2) = 4, T(15) = 28$$
$$8a + 4b + 2c + d = 4 \qquad (1)$$
$$3375a + 225b + 15c + d = 28 \qquad (2)$$
$$\frac{dT}{dt} = 3at^2 + 2bt + c$$
$$\frac{dT}{dt}(2) = 0 = \frac{dT}{dt}(15)$$
$$12a + 4b + c = 0 \qquad (3)$$
$$675a + 30b + c = 0 \qquad (4)$$

From calculator solver for simultaneous equations:

$a = -0.022, b = 0.56, -2.0, d = 5.9$
$T = -0.022t^3 + 0.56t^2 - 2.0t + 5.9$

b According to the model, $T(23.8) = -7.97°$ C

The model proposes a temperature for midnight at end of day as $-9.84°$ C, which is far from the 5.9° C predicted for midnight at the start of the day. The model therefore won't be viable to track consecutive days.

22 $15 \leqslant a < 25$ and $21.5 \leqslant b < 22.5$

$f'(x) = 48 - 2x$ so the maximum is at $x = 24$; the function is strictly increasing for $21 \leqslant x < 22.5$.

$f(a)$ has range $f(15) \leqslant f(a) < f(24): -77 \leqslant f(a) \leq 4$

$f(b)$ has range $f(21.5) \leqslant f(b) < f(22.5): -2.25 \leqslant f(b) < 1.75$

Then the maximum value of $f(a) = f(b)$ is $4 - (-2.25) = 6.25$

23 **a** Estimating using vertical strips for $7 \leq x < 12$

Each strip has the equivalent approximate area to a trapezoid with side lengths being the difference of the two ends. The first strip is a triangle which can be considered as a trapezoid with one side length being zero.

Trapezoidal rule approximation:

$x_0 = 7, n = 5, h = 1$

x	y	$\times$	$=$
$x_0 = 7$	0	$\times 1$	0
$x_1 = 8$	$9 - 6 = 3$	$\times 2$	6
$x_2 = 9$	$11 - 4 = 7$	$\times 2$	14
$x_3 = 10$	$12 - 3 = 9$	$\times 2$	18
$x_4 = 11$	$12 - 3 = 9$	$\times 2$	18
$x_5 = 12$	$13 - 3 = 10$	$\times 1$	10
	TOTAL		66
	TOTAL $\times \frac{h}{2}$		33

Approximate area: 33 square units.

b Area on map is $33\ \text{cm}^2$.

This is equivalent to an actual area of $33 \times 25^2 = 20\,625\ \text{m}^2$.

Core Review Exercise

1 a

$$u_1 = 7$$
$$u_3 = 15 = u_1 + 2d$$
$$2d = 8$$
$$d = 4$$

b $u_{20} = u_1 + 19d = 83$

c

$$S_n = \frac{n}{2}(u_1 + u_n)$$
$$S_{20} = \frac{20}{2}(7 + 83) = 900$$

2 a $y = \frac{2x-12}{3} = \frac{2}{3}x - 4$

Gradient of L is $\frac{2}{3}$

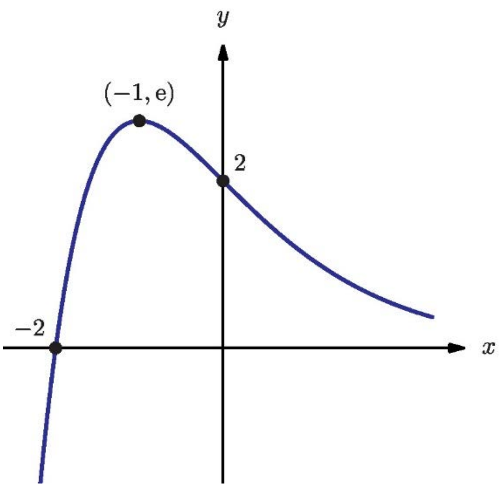
b $k = y(9) = \frac{2}{3}(9) - 4 = 2$

c M has gradient $-\frac{3}{2}$, passes through $(9, 2)$.

M has equation $y - 2 = -\frac{3}{2}(x - 9)$

$$2y - 4 = -3x + 27$$
$$3x + 2y = 31$$

3 a From GDC:



b Horizontal asymptote $y = 0$

c $\text{f}(x) \leqslant \text{e}$

4 **a** $a = \frac{5}{2} = 2.5$

The domain is $x \geqslant 2.5$

b Range is $f(x) \geqslant 0$

c

$$f(x) = \sqrt{x - \frac{5}{2}}$$

$$f^{-1}(x) = x^2 + \frac{5}{2}$$

$$f^{-1}(4) = \frac{37}{2} = 18.5$$

5 **a** Using cosine rule:

$$\cos A\hat{C}B = \frac{AC^2 + BC^2 - AB^2}{2(AC)(BC)} = 0.778$$

$$A\hat{C}B = 38.9°$$

b

$$\text{Area} = \frac{1}{2}(AC)(CB)\sin A\hat{C}B$$

$$= 95.5 \text{ cm}^2$$

6 $X \sim \mathrm{N}(26, 20.25)$

a $\sigma = \sqrt{20.25} = 4.5$

b From GDC, $\mathrm{P}(21.0 < x < 25.3) = 0.305$

c From GDC, $a = 28.2$

7 **a** $y = (2 - x)(2 + x)e^{-x}$ has roots at $A(-2,0)$ and $B(2,0)$

b From GDC:

$$\text{Shaded area} = \int_{-2}^{2} (4 - x^2)e^{-x}\, dx = 15.6$$

8 **a** From GDC: Median is 4.5

b Lower quartile: 3

Upper quartile: 5

$\mathrm{IQR} = 5 - 3 = 2$

c $n = 10$

Number of students scoring at least 4: 7

$$\mathrm{P}(X \geqslant 4) = \frac{7}{10}$$

Worked solutions

Worked solutions

9 **a**

$$\text{Volume} = \frac{1}{3}(\text{Base area}) \times \text{height}$$
$$= \frac{1}{3} \times 3.2^2 \times 2.8$$
$$= 9.56 \text{ cm}^3$$

b Mass $= 9.3 \times 9.56 = 88.9$ g

c Considering triangle ABC:

$$AC^2 = AB^2 + BC^2 = 2 \times 3.2^2$$

So $AO^2 = \frac{3.2^2}{2}$

Considering triangle AOV:

$$AV = \sqrt{AO^2 + OV^2}$$
$$= \sqrt{\frac{3.2^2}{2} + 2.8^2}$$
$$= 3.6 \text{ cm}$$

d Using cosine rule in triangle BVC:

$$\cos B\hat{V}C = \frac{BV^2 + VC^2 - BC^2}{2(BV)(VC)} = 0.605$$

$$B\hat{V}C = 52.8°$$

e Each triangular side has area

$$\triangle \text{ Area} = \frac{1}{2}(BV)(VC)\sin\left(B\hat{V}C\right) = 5.16 \text{ cm}^2$$

$$\text{Base area} = 3.2^2 = 10.2 \text{ cm}^2$$

$$\text{Total surface area} = 4(5.16) + 10.2 = 30.9 \text{ cm}^2$$

10 **a** Root at root of numerator: $x = -\frac{3}{2}$

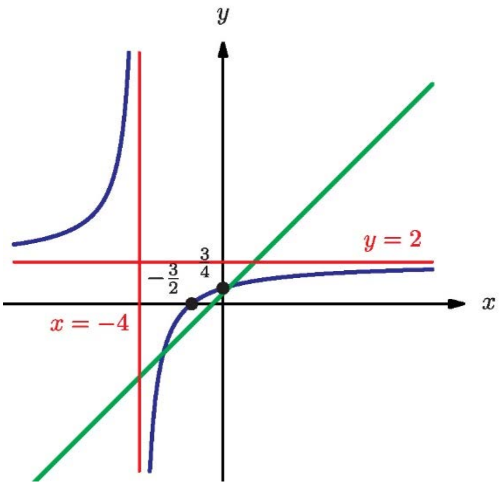
y-intercept at $x = 0$: $\left(0, \frac{3}{4}\right)$

Vertical asymptote at root of denominator: $x = -4$

Horizontal asymptote is value as $x \to \pm\infty$: $y = 2$

b $x = -4$

c



d $(-2.85078, -2.35078)$ or $(0.35078, 0.85078)$

e 1

f L has gradient -1 and passes through $(-2, -3)$

L has equation $y + 3 = -(x + 2)$

$$y = -x - 5$$

11

$$y(-2) = -2 + 3(-2)^2 = 10$$
$$\frac{dy}{dx} = 1 + 6x$$
$$\frac{dy}{dx}(-2) = -11$$

Normal at $(-2,10)$ has gradient $\frac{1}{11}$

Normal has equation $y - 10 = \frac{1}{11}(x + 2)$

$$11y - 110 = x + 2$$
$$x - 11y + 112 = 0$$

12 a $\log_{10} x = 3$ so $x = 10^3 = 1000$

b $\log_{10} 0.01 = -2$

13 a The question has been corrected to: Expand and simplify $(2x^2)^3(x - 3x^{-5})$

$(2x^2)^3(x - 3x^{-5}) = 8x^6(x - 3x^{-5}) = 8x^7 - 24x^{-2}$

b $\frac{d}{dx}(2x^2)^3(x - 3x^{-5}) = 56x^6 + 48x^{-3}$

14 a $u_4 = 18r^3$

b $S_{15} = \frac{18(1-r^{15})}{1-r}$

c $\frac{18(1-r^{15})}{1-r} = 26.28$

From GDC: $r = -1.05$ or 0.315

Try computing the sum using r=-1.05, and you will find that the only valid solution is r=0.315

15 Assuming monthly compounded interest

Using GDC:

$n = 18$

$I\% = ?$

$PV = 200$

$PMT = 0$

$FV = -211$

$P/Y = 12$

$C/Y = 12$

Solving for $I\%$: $I\% = 3.57$

He needs an annual interest rate of 3.6% (to 1 d.p.)

16 Require that

$$\sum \mathrm{P}(X = x) = 1$$

$$k\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\right) = \frac{205}{144}$$

$$k = \frac{144}{205}$$

$$\begin{aligned}\mathrm{E}(X) &= \sum x\,\mathrm{P}(X = x)\\ &= k\left(\frac{1}{1^2} + \frac{2}{2^2} + \frac{3}{3^2} + \frac{4}{4^2}\right)\\ &= \frac{25}{12}k\\ &= \frac{300}{205}\\ &= \frac{60}{41}\end{aligned}$$

17 Let X be the number of brown eggs in a box.

$X \sim \mathrm{B}(240, 0.05)$

a $\mathrm{E}(X) = 240 \times 0.05 = 12$

b $\mathrm{P}(X = 15) = 0.0733$

c

$$\begin{aligned}\mathrm{P}(X \geqslant 10) &= 1 - \mathrm{P}(X \leqslant 9)\\ &= 1 - 0.236\\ &= 0.764\end{aligned}$$

18 a Gradient $CD = \frac{1-(-1)}{-2-(-1)} = -2$

b Gradient $AD = \frac{1-(-1)}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$

Since $-2\left(\frac{1}{2}\right) = -1$, the two lines are perpendicular

c CD has gradient -2 and passes through $D(-1,-1)$

CD has equation $y+1=-2(x+1)$

$y=-2x-3$ (1)

d AB has equation $x+3y=6$ (2)

Finding the intersection point:

$(1)-2(2):-5y=-15$

$y=3$ so $x=6-3y=-3$

The intersection point is $E(-3,3)$

e $AD=\sqrt{(3-(-1))^2+(1-(-1))^2}=\sqrt{20}$

f Since $AD\perp ED$:

Area $ADE=\frac{1}{2}(AD)(ED)=\frac{1}{2}\sqrt{20}\sqrt{20}=10$

19 a i

$$\begin{aligned}\text{Volume} &= \pi r^2 h\\ &= \pi(3.25)^2\times 39\\ &= 1294.14\text{ cm}^2\end{aligned}$$

ii The diameter is 6.5 cm so the maximum number of balls is the rounded down value of $\frac{39}{6.5}=6$

iii I Total volume occupied by the balls:

$$\begin{aligned}\text{Ball volume} &= 6\times\frac{4}{3}\pi r^3\\ &= 8\pi(3.25)^3\\ &= 862.76\text{ cm}^3\end{aligned}$$

Then the volume of air is $1294.14-862.76=431.38\text{ cm}^3$

II This is equal to $431.38\times10^{-6}=4.31\times10^{-4}\text{ m}^3$

b i I $B\hat{T}L+B\hat{L}T+T\hat{B}L=180°$ so $B\hat{T}L=180-80-26.5=73.5°$

II Using the sine rule:

$$\begin{aligned}BT &= \frac{BL}{\sin B\hat{T}L}\times\sin B\hat{L}T\\ &= 55.8\text{ m}\end{aligned}$$

ii $TG=BT\sin 80°=55.0\text{ m}$

iii Using cosine rule in triangle BTM:

$$\begin{aligned}TM^2 &= BM^2+BT^2-2(BM)(BT)\cos T\hat{B}M\\ &= 200^2+55.8^2-2(200)(55.8)\cos 100°\\ &= 46997\end{aligned}$$

$TM=217\text{ m}$

20 a i $r = 0.985$

ii This is a strong positive correlation; as number of bicycles increases, production cost reliably increases in a mostly linear pattern.

b From GDC:

$$y = 260x + 699$$

c Using the regression equation, $\hat{y}(13) = \$4\,079$

d The total income for those 13 bicycles is $13 \times 304 = \$3\,952$ which is less than the estimated cost of production given in part **c**.

e i $\text{Sale price}(x) = 304x$

ii $\text{Profit} = 304x - (260x + 699) = 44x - 699$

iii For positive profit, require $44x > 699$

$$x > 15.8$$

For positive profit, the factory must produce at least 16 bicycles per day.

21 Let X be the time taken (in minutes) to complete a test paper.

$$X \sim \text{N}(52, 7^2)$$

a From GDC: $\text{P}(X < 45) = 0.159$

b Let Y be the number of students, in a group of 20, who complete the test in less than 45 minutes.

$$Y \sim \text{B}(20, 0.159)$$

i $\text{P}(Y = 1) = 0.119$

ii

$$\begin{aligned}\text{P}(Y > 3) &= 1 - \text{P}(Y \leq 3)\\ &= 1 - 0.606\\ &= 0.394\end{aligned}$$

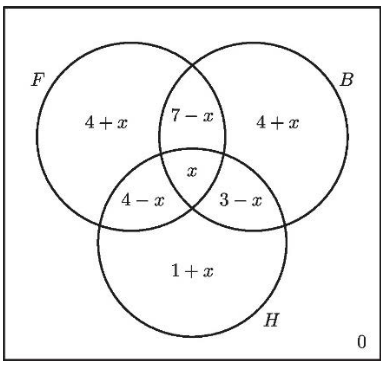
22 If x study all three subjects (and every student studies at least one):

$7 - x$ study French and Biology but not History

$4 - x$ study French and History but not Biology

Then the number studying French only must be $15 - (7 - x + 4 - x + x) = 4 + x$

Working similarly:



a The total number of students displayed in the diagram is

$$4 + x + 7 - x + 4 - x + x + 4 + x + 3 - x + 1 + x = 23 + x = 26$$

So $x = 3$

b $\mathrm{P}(F \cap B' \cap H') = \frac{7}{26}$

c $\mathrm{P}(B'|F) = \frac{\mathrm{P}(B' \cap F)}{\mathrm{P}(F)} = \frac{8}{15}$

d A total of 8 students study History, so 18 do not.

The probability that neither of two randomly selected students study History is

$$P(H', H') = \frac{18}{26} \times \frac{17}{25} = \frac{153}{325}$$

Then the probability that at least one studies history is the complement of this:

$$\mathrm{P}(\text{at least one } H) = 1 - \frac{153}{325} = \frac{172}{325}$$

Worked solutions

Applications and interpretation: Practice Paper 1

Solutions

1 **a** 13 cm

b 38 (any answer between 30 and 40 is reasonable)

c $\frac{125}{200} = \frac{5}{8} = 0.625$

2 **a** From GDC: minimum value is at $x = 0$, so range is

$$f(x) \geqslant \ln\left(\frac{1}{3}\right) = -1.10$$

b $f(1) = -0.41$

c From GDC: $f(x) = 5$ for $x = \pm 21.1$

3 **a**

$$u_{10} = a + 9d = 285 \quad (1)$$

$$S_{25} = \frac{25}{2}(2a + 24d) = 9000$$

$$25a + 300d = 9000 \quad (2)$$

b)

$(2) - 25(1): 75d = 1875$

$d = 25$ so $a = 285 - 9d = 60$

4 **a** 80

b $P(1 \cap \mathrm{HL}) = \frac{10}{80} = 0.125$

c $P(\mathrm{SL}|2) = \frac{30}{50} = 0.6$

5 **a** Using GDC:

$n = 60$

$I\% = 12\%$

$PV = 20\,000$

$PMT = ?$

$FV = 0$

$P/Y = 12$

$C/Y = 12$

Solving for PMT: $PMT = -444.89$

She must make monthly payments of AUS 444.89 per month.

b The total payments she would make total $60 \times 444.89 = 26\,693.40$

Therefore the total amount she would pay in interest is AUS 6 693.40

6 **a** H_0: Wine preference is independent of gender.

H_1: Wine preference and gender are not independent.

b Degrees of freedom: $(3-1) \times (2-1) = 2$

$\chi^2 = 4.74 \quad p = 0.0935 > 0.05$

c $p > 0.05$ so do not reject H_0; there is insufficient evidence at 5% level to conclude that gender and wine preference are not independent.

7 **a** $\theta = 60°$

$$\text{Sector area} = \frac{\pi\theta r^2}{360} = \frac{\pi r^2}{6}$$

$$\text{Volume} = \text{Sector area} \times \text{height} = \frac{\pi r^2 h}{6}$$

$$k = 6$$

b

$$13.5 \leqslant r < 14.5$$
$$8.45 \leqslant h < 8.55$$
$$\frac{\pi(13.5)^2(8.45)}{6} \leqslant V < \frac{\pi(14.5)^2(8.55)}{6}$$
$$806\text{ cm}^3 \leqslant V < 941\text{ cm}^3$$

8 **a** **i** $D(0) = 0.5$ so initially there are 500 infected trees.

ii $D(2) = 4.220$ so after 2 years there are 4220 infected trees.

b From calculator, $D = 6$ when $t = 3.48$

There will be 6000 trees after 3.48 years (about a week less than 3 years and 6 months)

c The model predicts that the number of diseased trees will increase towards an upper limit of 10 000, which may represent the total number of trees in this population.

9 **a** Let the base of the pyramid be $ABCD$ and the vertex E, with the midpoint of the base X lying vertically below E so that $EX = 50$ cm

Then triangle AXE has angle $A\hat{X}E = 90°$ and $X\hat{A}E = 75°$

By trigonometry $AE = \frac{50}{\sin 75°} = 51.8$ cm

b And $AX = \frac{50}{\tan 75°} = 13.4$ cm

Then the base diagonal $AC = 2AX = 26.8$ cm

c The base area is given by $\frac{AC^2}{2} = 359.12\text{ cm}^2$

$$\text{Volume} = \frac{1}{3}(\text{base area}) \times \text{height}$$
$$= \frac{1}{3} \times 359.12 \times 50$$
$$= 5\,985\text{ cm}^3$$

10 a $H_0: \mu_A = \mu_B$

$H_1: \mu_A \neq \mu_B$

b From GDC, using pooled variance: $p = 0.0338 < 0.1$

c $p < 0.1$ so reject H_0; there is sufficient evidence at the 10% level that the mean length of perch in the two river populations is different.

d It was assumed that both populations were normally distributed and also that they have the same variance.

11 a The edge which is the perpendicular bisector of BC:

$$\text{Gradient } BC = \frac{6-2}{6-3} = \frac{4}{3}$$

$$\text{Gradient of edge } \perp_{BC} = -\frac{3}{4}$$

Midpoint of BC is $(4.5, 4)$

Equation of edge $\perp_{BC}$ is $y - 4 = -\frac{3}{4}(x - 4.5)$

$4y - 16 = -3x + 13.5$

$3x + 4y = 29.5$

$6x + 8y = 59$

b Drawing the additional line on the diagram, $(4, 5)$ is still in the cell of C, so the phone mast providing the signal in $(4, 5)$ will be C.

c State at which point someone must be to potentially be receiving a signal from mast A, C and E.

The vertex $(6, 11)$ receives signal from A, C and E.

12 a $P(0) = -3.5$ so $c = -3.5$

$P(7) = 0 = 49a + 7b - 3.5$ (1)

By symmetry around the maximum $P(4.5)$:

$P(2) = 0 = 4a + 2b - 3.5$ (2)

Using calculator to solve:

$a = -0.25, b = 2.25$

b Then $P(4.5) = 1.56$

Her maximum weekly profit is £1560.

13 a

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{k}{\sqrt{t}}$$

$$\frac{\mathrm{d}l}{\mathrm{d}t}(4) = 0.14 = \frac{k}{\sqrt{4}}$$

$$k = 0.28$$

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{0.28}{\sqrt{t}} = 0.28t^{-0.5}$$

b

$$l = \int \frac{\mathrm{d}l}{\mathrm{d}t}\,\mathrm{d}t + l(0)$$

$$= 0.56t^{0.5} + 1.2$$

So $l(25) = 4$

A 25 year old shark is predicted by the model to be 4 m long.

c The model indicates that sharks always have a positive growth – they never stop growing – but that growth slows as the shark ages.

Worked solutions

Applications and interpretation: Practice Paper 2

Worked solutions

Solutions

1 a i

$$y = \frac{12 - 3x}{2} = 6 - \frac{3}{2}x$$

Gradient of l_1 is $-\frac{3}{2}$

ii $N(0,6)$

b l_2 has gradient $\frac{2}{3}$ and passes through $P(3, -5)$

l_2 has equation $y + 5 = \frac{2}{3}(x - 3)$

$3y + 15 = 2x - 6$

$2x - 3y - 21 = 0$

c

$3x + 2y = 12 \quad (1)$

$2x - 3y = 21 \quad (2)$

Intersection:

$3(1) + 2(2): 13x = 78$

$x = 6$

Q has coordinates $(6, -3)$

d i $NQ = \sqrt{(6-0)^2 + (-3-6)^2} = \sqrt{117} \approx 10.8$

ii $PQ = \sqrt{(6-3)^2 + (-3+5)^2} = \sqrt{13} \approx 3.61$

e Since $NP \perp PQ$, area $NPQ = \frac{1}{2}(NP)(PQ) = 19.5$

2 a

$$\sum \mathrm{P}(X = x) = 1 = 0.73 + p + q$$

$p + q = 0.27 \quad (1)$

b

$$E(X) = \sum xP(X = x)$$
$$= 2(0.04) + 4(0.07) + 6p + 8q + 10(0.62)$$
$$= 6.56 + 6p + 8q = 8.56$$

$6p + 8q = 2$

$3p + 4q = 1$ (2)

c

$(2) - 3(1): q = 0.19$

So $p = 0.27 - q = 0.08$

d Let Y be the number of times, in 15 attempts, that Michele hits a bullseye

$Y \sim B(15,0.62)$

i $P(Y = 8) = 0.161$

ii

$$P(Y \geqslant 10) = 1 - P(Y \leqslant 9)$$
$$= 1 - 0.533$$
$$= 0.467$$

e **i** $E(Y) = 15 \times 0.62 = 9.3$

ii $\text{var}(Y) = 15 \times 0.62 \times (1 - 0.62) = 3.534$

$\text{sd}(Y) = \sqrt{\text{var}(Y)} = 1.88$

f Let W be the number of rounds in which Michele hits the bullseye at least 10 times.

$W \sim B(5, 0.467)$

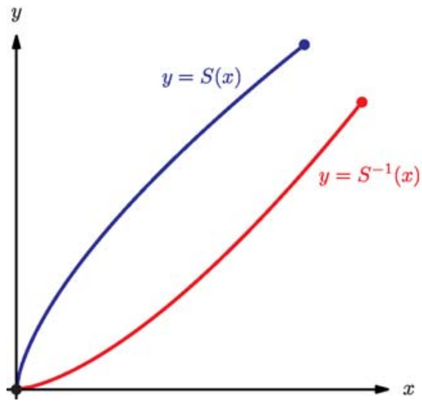
$$P(W > 3) = 1 - P(W \leqslant 3)$$
$$= 1 - 0.851$$
$$= 0.149$$

3 **a** A cube with edge length x has 6 faces, each with area x^2 and volume $V = x^3$.

Then $S = 6x^2 = 6V^{\frac{2}{3}}$

$k = 6, c = \frac{2}{3}$

b **i**



ii The graph of the inverse is the same curve, reflected through the line $y = x$.

c **i** The domain of S is $0 \leqslant V \leqslant 125$ so the range of S^{-1} is $0 \leqslant S^{-1} \leqslant 125$

ii The range of S is $0 \leqslant S \leqslant 150$ so the domain of S^{-1} is $0 \leqslant S \leqslant 150$

d $S^{-1}(x) = \left(\frac{x}{6}\right)^{\frac{3}{2}}$

i $S^{-1}(54) = 27$

ii $S^{-1}(54)$ gives the volume of a cube when the surface area is 54 square units.

4 **a** $d(0) = d(8) = 0$ and $d(4) = 2.3$

The half-period of the sine wave must be 8 so $q = \frac{360}{8} = 22.5$

The amplitude of the sine wave must be 2.3 so $p = 2.3$

b $d(2) = 2.3 \sin 45° = 1.6263 \ldots = 1.63$ m

c

$$\begin{aligned}\text{Percentage error} &= \frac{|\text{True value} - \text{Approximate value}|}{\text{True value}} \times 100\% \\ &= \frac{2.5 - 1.6263 \ldots}{2.5} \times 100\% \\ &= 34.9\%\end{aligned}$$

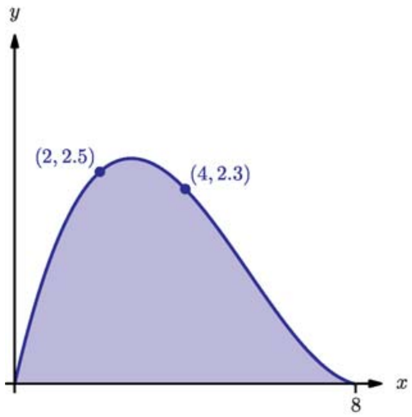
d Using the values $d(0) = d(8) = 0, d(4) = 2.3, d(2) = 2.5$:

$$\begin{cases} 512a + 64b + 8c = 0 \\ 64a + 16b + 4c = 2.3 \\ 8a + 4b + 2c = 2.5 \end{cases}$$

Using calculator solver: $a = 0.0323, b = -0.531, c = 2.18$

Worked solutions

e



f **i** From GDC, maximum d is $d(2.735\ldots) = 2.65$ m

ii

$$\text{From GDC: } \int_0^8 d(x)\ \mathrm{d}x = 12.2\ \mathrm{m}^2$$

5 **a** $r = 0.978$

b r is very close to 1, indicating a very strong positive correlation; the points lie nearly on a straight line with positive gradient, so a linear model is suitable.

c From GDC, after sorting the data:

$a = 0.0444, b = 2.56, c = 0.0152, d = 8.40$

d **i** $90a + b = 6.55$

Predict a cost of 6550 Euros

ii $200a + b = 11.4$

Predict a cost of 11 400 Euros

iii $330c + d = 13.4$

Predict a cost of 13 400 Euros

e **i** is reliable as 90 lies within the range of the data for the straight line.

ii is fairly reliable; 200 is actually outside the range of the data for the regression line $ax + b$, but the whole data set is known to be strongly linear, and 200 does lie within the complete data set.

iii is unreliable, as 330 lies outside the data set altogether, and extrapolation is not reliable.

6 **a** **i**

$$y = x^{-1}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -x^{-2}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}(p) = -\frac{1}{p^2}$$

ii Then the tangent at (p, p^{-1}) will have gradient $-p^{-2}$ and so will have equation

$$y - p^{-1} = -p^{-2}(x - p)$$
$$y - p^{-1} = -p^{-2}x + p^{-1}$$
$$p^2y - 2p = -x$$
$$x + p^2y - 2p = 0$$

b i On the tangent line:

When $x = 0, y = \frac{2}{p}$ so R has coordinates $\left(0, \frac{2}{p}\right)$

When $y = 0, x = 2p$ so Q has coordinates $(2p, 0)$

ii Then area $OQR = 0.5(2p)(2p^{-1}) = 2$, independent of the value of p.

c $QR = \sqrt{(2p^{-1})^2 + (2p)^2} = 2\sqrt{p^2 + p^{-2}}$

d

Tip: Find minimum using GDC or calculus method such as shown below.

Remember that when finding a minimum distance, where the distance has been given by a formula including a square root, it is usually easier to differentiate the square of the distance. Since distance is non-negative, the minimum of the distance and the minimum of its square must occur at the same value of the variable.

QR will have a minimum when QR^2 has a minimum

$$QR^2 = 4p^2 + 4p^{-2}$$
$$\frac{\mathrm{d}}{\mathrm{d}p}(QR^2) = 8p - 8p^{-3} = 8p^{-3}(p^4 - 1)$$
$$\frac{\mathrm{d}}{\mathrm{d}p}(QR^2) = 0 \text{ when } p^4 = 1, \text{so } p = 1 \text{ (since } p > 0)$$

This represents a minimum since $\frac{\mathrm{d}}{\mathrm{d}p}(QR^2) < 0$ for $0 < p < 1$ and $\frac{\mathrm{d}}{\mathrm{d}p}(QR^2) > 0$ for $p > 1$.

Minimum $QR = 2\sqrt{1+1} = 2\sqrt{2} \approx 2.83$